



**Princeton University - Economic Theory Center
Research Paper No. 32-2012**

Recursive Utility in a Markov Environment with Stochastic Growth

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**BFI Working Paper Series
No. 2012-002**

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Abstract

Recursive utility models of the type introduced by Kreps and Porteus (1978) are used extensively in applied research in macroeconomics and asset pricing in environments with uncertainty. These models represent preferences as the solution to a nonlinear forward-looking difference equation with a terminal condition. Such preferences feature investor concerns about the intertemporal composition of risk. In this paper we study infinite horizon specifications of this difference equation in the context of a Markov environment. We establish a connection between the solution to this equation and to an arguably simpler Perron-Frobenius eigenvalue equation of the type that occurs in the study of large deviations for Markov processes. By exploiting this connection, we establish existence and uniqueness results. Moreover, we explore a substantive link between large deviation bounds for tail events for stochastic consumption growth and preferences induced by recursive utility.

Keywords: recursive utility, Markov process, stochastic growth, large deviations

*We benefitted from presenting this work at the Stanford SITE conference in 2011, from discussions with Henri Berestycki, Anmol Bhandari, Xiaohong Chen, Valentin Haddad, Eric Renault, and from discussions with and computational assistance from Mark Hendricks.

1 Introduction

Recursive utility models of the type suggested by Kreps and Porteus (1978) and featured in the asset pricing literature by Epstein and Zin (1989) and others represent preferences as the solution to a nonlinear forward-looking difference equation with a terminal condition. Such preferences are used in economic dynamics because seemingly simple parametric versions provide a convenient device to change risk aversion while maintaining the same intertemporal elasticity of intertemporal substitution. In this paper we explore infinite horizon specifications in the context of a Markov environment. Even under the Markov specification, establishing the existence of a solution to this forward-looking recursion used to depict preferences can be challenging.¹ In this paper we establish a connection between the solution to this equation and to an arguably simpler eigenvalue equation of the type that occurs in the study of large deviations for Markov processes, see Donsker and Varadhan (1975), Donsker and Varadhan (1976), Balaji and Meyn (2000) and Kontoyiannis and Meyn (2005).

The remainder of the paper is organized as follows. In section 2 we state formally the recursive utility problem and a related Perron-Frobenius eigenvalue problem. In section 3 we use the the latter problem to construct a change in probability that will play a central role in our analysis. Under this change of measure, we establish several inequalities in section 4. Section 5 states and proves our main analytical result, and section 6 expands on some of the ramifications our analysis. Finally, in section 7 we discuss the link to the analysis of large deviations applied to a Markov process.

2 Two related problems

Consider a discrete-time specification of recursive preferences of the type suggested by Kreps and Porteus (1978) and Epstein and Zin (1989). We use the homogeneous-of-degree-one aggregator specified in terms of current period consumption C_t and the continuation value V_t :

$$V_t = (\zeta C_t)^{1-\rho} + \exp(-\delta) [\mathcal{R}_t(V_{t+1})]^{1-\rho} \frac{1}{1-\rho}. \quad (1)$$

¹See Marinacci and Montrucchio (2010) for a recent thorough analysis of existence and uniqueness of continuation value processes, but the sufficient conditions given there impose restrictions that preclude some of the parametric models used in practice.

where

$$\mathcal{R}_t(V_{t+1}) = E (V_{t+1})^{1-\gamma} | \mathcal{F}_t^{\frac{1}{1-\gamma}}$$

adjusts the continuation value V_{t+1} for risk. With these preferences, $\frac{1}{\rho}$ is the elasticity of intertemporal substitution and δ is a subjective discount rate. Finally, the parameter ζ does not alter preferences, but it gives some additional flexibility, and we will select it in a judicious manner.

Next exploit the homogeneity-of-degree one specification of the aggregator (1) and divide by C_t to obtain:

$$\frac{V_t}{C_t} = \zeta^{1-\rho} + \exp(-\delta) \mathcal{R}_t \left(\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \right)^{\frac{1}{1-\rho}}. \quad (2)$$

Applying the aggregator requires a terminal condition for the continuation value. In what follows we will consider infinite-horizon limits. Thus we will explore the construction of the continuation value V_t as a function of $C_t, C_{t+1}, C_{t+2}, \dots$. For convenience we rewrite (2) as

$$\left(\frac{V_t}{C_t} \right)^{1-\rho} = \zeta^{1-\rho} + \exp(-\delta) \mathcal{R}_t \left(\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \right)^{1-\rho}$$

Consider now a Markov specification in discrete time. Let $(X, Y) = \{(X_t, Y_t)\}$ be an underlying Markov process, and suppose that

Assumption 2.1. (a) *The joint distribution of (X_{t+1}, Y_{t+1}) conditioned on (X_t, Y_t) depends only on X_t .*

(b) *Consumption dynamics evolve as:*

$$\log C_{t+1} - \log C_t = \kappa(X_{t+1}, Y_{t+1}, X_t).$$

When the joint process (X, Y) is stationary, the logarithm of consumption has stationary increments and the level process for consumption displays stochastic geometric growth. For convenience we normalize $C_0 = 1$. Given our assumed homogeneity in preference, it is straightforward to allow for more general initial conditions.² This specification allows us to feature the the process X in our analysis while allowing for some additional flexibility.

²In the special case in which κ does not depend on Y_{t+1} , the consumption process is what is called a *multiplicative functional in the applied mathematics literature*.

Generally, we may think of this as a convenient specification of consumption that could emerge from a model in which consumption is determined endogenously.

Given the Markov dynamics, we seek a solution:

$$\frac{V_t}{C_t}^{1-\rho} = \hat{f}(X_t).$$

for $f \geq 0$. Thus equation (2) can be expressed equivalently as

$$\hat{f}(x) = \zeta^{1-\rho} + \exp(-\delta) E^h \hat{f}(X_{t+1})^\alpha \exp[(1-\gamma)\kappa(X_{t+1}, Y_{t+1}, X_t)] | X_t = x \quad (3)$$

where

$$\alpha = \frac{1-\gamma}{1-\rho}.$$

We write this equation compactly as:

$$\hat{f}(x) = \mathbb{U}\hat{f}(x) \quad (4)$$

where $\mathbb{U}(f)$ is given by the right-hand side of (3). Notice that

$$\mathbb{U}f(x) \geq \zeta^{1-\rho}. \quad (5)$$

An alternative equation will also be of interest. Construct this equation using:

$$\frac{V_t}{C_t}^{1-\gamma} = \bar{f}(X_t)$$

In this case the fixed-point equation of interest is

$$\bar{f}(x) = \zeta^{1-\rho} + \exp(-\delta) E^h \bar{f}(X_{t+1}) \exp[(1-\gamma)\kappa(X_{t+1}, Y_{t+1}, X_t)] | X_t = x \quad (6)$$

which we obtain by raising both sides of (3) to the power α . We write this equation compactly as:

$$\bar{f}(x) = \mathbb{V}\bar{f}(x) \quad (7)$$

where $\mathbb{V}(f)$ is given by the right-hand side of (6). Notice that

$$(\mathbb{V}f)^\alpha = \mathbb{U} f^\alpha .$$

In particular, there is a one-to-one correspondence between positive fixed points of \mathbb{V} and positive fixed points of $\mathbb{U}f$. Furthermore,

$$\mathbb{V}f(x) \geq \zeta^{1-\gamma} \tag{8}$$

provided that $\alpha > 0$.

Remarkably, the solution to the fixed point equations (4) and (6) are closely related to a Perron-Frobenius eigenvalue equation of the type analyzed by Hansen and Scheinkman (2009) in their study of risk-return relations and risk pricing over long-term investment horizons. The eigenvalue problem studied in Hansen and Scheinkman (2009) is also closely related to an eigenvalue equation that occurs in the study of large-deviations. The eigenvalue equation is:

$$E [\exp[(1 - \gamma)\kappa(X_{t+1}, Y_{t+1}, X_t)]e(X_{t+1})|X_t = x] = \exp(\eta)e(x) \tag{9}$$

for $e > 0$. In many specifications this equation has multiple positive solutions with eigenfunctions that are not equal up to a scale factor.

3 Changing the probability measure

We use a Perron-Frobenius eigenfunction to change the probability measure. Associated with each such eigenfunction is a positive random variable

$$M_{t+1} = \frac{\exp[(1 - \gamma)\kappa(X_{t+1}, Y_{t+1}, X_t)]e(X_{t+1})}{e(X_t)} \exp(-\eta)$$

which has conditional expectation equal to unity. We use this to define a change of measure for the transition probability of the Markov process, via:

$$\hat{E} [\phi(X_{t+1}, Y_{t+1})|X_t = x, Y_{t+1} = y] = E [M_{t+1}\phi(X_{t+1}, Y_{t+1})|X_t = x]$$

for any Borel measurable function ϕ with the appropriate domain. This change in the transition probability preserves the Markov property and the restrictions imposed by Assumption 2.1. Only one of the eigenfunctions induces a change of measure that is stochastically

stable in the sense that³

Assumption 3.1. *Under the change of probability measure,*

$$\lim_{t \rightarrow \infty} \mathbb{E}[\phi(X_t, Y_t) | X_0 = x] = \mathbb{E}[\phi(X_t, Y_t)]$$

for any bounded Borel measurable function ϕ . The expectation on the right-hand side uses a stationary distribution implied by the change in the transition distribution. We require that the convergence apply for almost all Markov states (x, y) under the stationary distribution for the change in measure.

There is an extensive literature that gives sufficient conditions for stochastic stability.

To apply this change in measure, we use a multiplicative scaling of functions:

$$g(x) = f(x)e(x)^{-\frac{1}{\alpha}}.$$

Notice that

$$\begin{aligned} & \exp(-\delta) (E[f(X_{t+1})^\alpha \exp[(1-\gamma)\kappa(X_{t+1}, Y_{t+1}, X_t)] | X_t = x])^{\frac{1}{\alpha}} \\ &= \exp(-\xi) e(x)^{\frac{1}{\alpha}} \mathbb{E}(g(X_{t+1})^\alpha | X_t = x)^{\frac{1}{\alpha}} \end{aligned}$$

where

$$\xi = \delta - \frac{\eta}{\alpha}$$

The transformed counterpart to recursion (4) is:

$$\hat{g}(x) = \mathfrak{U}\hat{g}(x)$$

where

$$\mathfrak{U}g(x) = \zeta^{1-\rho} e(x)^{\frac{-1}{\alpha}} + \exp(-\xi) \mathbb{E}[g(X_{t+1})^\alpha | X_t = x]^{\frac{1}{\alpha}}$$

This altered recursion has absorbed growth into a conveniently chosen change of measure.

In light of bound (5),

$$\mathfrak{U}g(x) \geq \zeta^{1-\rho} e(x)^{\frac{-1}{\alpha}} \tag{10}$$

³Uniqueness is established in Hansen and Scheinkman (2009) for a continuous-time Markov specification, but their result has a direct counterpart for discrete-time.

Next we consider the transformed counterpart to \mathbb{V} . In this case we let

$$h(x) = f(x)e(x)^{-1},$$

and notice that

$$\begin{aligned} & \exp(-\delta) (E [f(X_{t+1}) \exp[(1 - \gamma)\kappa(X_{t+1}, Y_{t+1}, X_t)] | X_t = x])^{\frac{1}{\alpha}} \\ &= \exp(-\xi)e(x)^{\frac{1}{\alpha}} \mathbb{E} [h(X_{t+1}) | X_t = x]^{\frac{1}{\alpha}}. \end{aligned}$$

The transformed counterpart to recursion (7) is:

$$\hat{h}(x) = \mathbb{V}\hat{h}(x)$$

where

$$\mathbb{V}\hat{h}(x) = \zeta^{1-\rho}e(x)^{-\frac{1}{\alpha}} + \exp(-\xi) \mathbb{E} [h(X_{t+1}) | X_t = x]^{\frac{1}{\alpha} \alpha}.$$

In light of bound (8),

$$\mathbb{V}\hat{h}(x) \geq \zeta^{1-\gamma}e(x)^{-1}$$

provided that $\alpha > 0$.⁴

The change-probability motivates one of our restrictions. To maintain discounting in the presence of stochastic growth, we assume:

Assumption 3.2. $\xi > 0$.

In terms of the initial parameters, the restriction on ξ implies that:

$$\delta > \frac{1 - \rho}{1 - \gamma} \eta$$

The scalar η is negative for typical parameterizations. Thus when $\gamma > 1$, this bound on δ is positive for $\rho < 1$ and negative when $\gamma > 1$.⁵

⁴Marinacci and Montrucchio (2010) construct L^∞ spaces weighted by scale factors that depend on time, including factors with geometric decay as a featured case. The L^∞ structure presumes processes with bounded support, although the support can increase over time because of the scale factors that they introduce. In contrast, we exploit heavily a Markov structure and use the Perron-Frobenius eigenvalue embedded in our change of probability measure to accommodate geometric growth and other convenient forms of stochastic growth in consumption. The recursion, \mathbb{U} , maps into special case of the recursions in Marinacci and Montrucchio (2010) for $\alpha < 0$ and $1 < \alpha \leq 1$; and the recursion, \mathbb{V} , maps into a special case when $\alpha > 0$ except that we feature L^1 spaces instead of L^∞ spaces.

⁵It is possible that η is positive, which alters the parameter restrictions.

4 Some Useful Inequalities

In this section we establish some useful inequalities that we will use to show the existence of fixed points to \mathfrak{U} and \mathfrak{V} . We will consider alternative operators with fixed points that are considerably easier to characterize. The fixed points of these operators will provide upper and lower bounds for the fixed points that interest us. Starting from these bounds, we will construct monotone sequences that converge point-wise to candidate fixed points of \mathfrak{U} and \mathfrak{V} . We also show when the constructed fixed-points coincide.

Recall that we have the flexibility to set $\zeta > 0$ in an arbitrary fashion. We exploit this convenience by setting ζ to satisfy:

$$\zeta^{1-\rho} = [1 - \exp(-\xi)].$$

4.1 Inequalities for $\hat{\mathfrak{U}}$

Suppose that $\alpha < 0$ and apply Jensen's inequality to obtain

$$\mathfrak{E}[g(X_{t+1})^\alpha | X_t = x] \geq \mathfrak{E}[g(X_{t+1}) | X_t = x]^\alpha. \quad (11)$$

Since $\frac{1}{\alpha} < 0$,

$$\mathfrak{E}[g(X_{t+1})^\alpha | X_t = x]^{\frac{1}{\alpha}} \leq \mathfrak{E}[g(X_{t+1}) | X_t = x]. \quad (12)$$

When $0 < \alpha < 1$, relation (11) holds with the reverse inequality and raising both sides to the $\frac{1}{\alpha}$ power preserves inequality (12). When $\alpha \geq 1$, relation (11) holds and raising both side to the power $\frac{1}{\alpha}$ gives us inequality (12) with the reverse sign . Thus we have

$$\begin{aligned} \mathfrak{U}g(x) &\leq \mathfrak{U}g(x) & \alpha < 0 \\ \mathfrak{U}g(x) &\leq \mathfrak{U}g(x) & 0 < \alpha < 1 \\ \mathfrak{U}g(x) &\geq \mathfrak{U}g(x) & \alpha \geq 1 \end{aligned}$$

where

$$\mathfrak{U}g(x) = [1 - \exp(-\xi)]e(x)^{-\frac{1}{\alpha}} + \exp(-\xi)\mathfrak{E}[g(X_{t+1}) | X_t = x].$$

A sufficient condition for the existence of a fixed point for \mathfrak{U} is:

Assumption 4.1. $\mathfrak{E} e(x)^{-\frac{1}{\alpha}} < \infty$.

A consequence of Assumption 4.1 is that a solution in L^1 (using the $\tilde{\cdot}$ stationary distri-

bution) to the fixed point problem $\mathfrak{U}\tilde{g} = \tilde{g}$ is given by

$$\tilde{g}(x) = [1 - \exp(-\xi)] \int_{t=0}^{\infty} \exp(-t\xi) \mathbb{E} [e(X_t)^{-\frac{1}{\alpha}} | X_0 = x]$$

since the right hand side converges in L^1 using the $\tilde{\nu}$ stationary distribution. In addition under Assumption 4.1, if $\alpha \leq 1$, since inequality (12) holds, \mathfrak{U} maps L^1_+ into L^1_+ .

4.2 Inequalities for \mathfrak{V}

Suppose again that $\alpha < 0$ and apply Jensen's inequality to obtain

$$\begin{aligned} & [1 - \exp(-\xi)]e(x)^{\frac{-1}{\alpha}} + \exp(-\xi) \mathbb{E} [h(X_{t+1}) | X_t = x]^{\frac{1}{\alpha}} \\ & \geq [1 - \exp(-\xi)]e(x)^{-1} + \exp(-\xi) \mathbb{E} [h(X_{t+1}) | X_t = x]^{\frac{1}{\alpha}} \end{aligned} \quad (13)$$

Raising both sides to the power α reverses the inequality and thus

$$\begin{aligned} & [1 - \exp(-\xi)]e(x)^{\frac{-1}{\alpha}} + \exp(-\xi) \mathbb{E} [h(X_{t+1}) | X_t = x]^{\frac{1}{\alpha}} \\ & \leq [1 - \exp(-\xi)]e(x)^{-1} + \exp(-\xi) \mathbb{E} [h(X_{t+1}) | X_t = x] \end{aligned}$$

For $0 < \alpha < 1$, the inequality in (13) remains the same and raising both side to power α does not reverse this inequality. For $\alpha \geq 1$ the inequality (13) is reversed and raising both sides to the power α does not reverse the inequality. Thus

$$\begin{aligned} \mathfrak{V}h(x) & \leq \mathfrak{V}h(x) & \alpha < 0 \\ \mathfrak{V}h(x) & \geq \mathfrak{V}h(x) & 0 < \alpha < 1 \\ \mathfrak{V}h(x) & \leq \mathfrak{V}h(x) & \alpha \geq 1 \end{aligned}$$

where

$$\mathfrak{V}h(x) = [1 - \exp(-\xi)]e(x)^{-1} + \exp(-\xi) \mathbb{E} [h(X_{t+1}) | X_t = x].$$

A sufficient condition for the existence of a fixed point for \mathfrak{V} is:

Assumption 4.2. $\mathbb{E}[e(x)^{-1}] < \infty$.

The fixed point for \mathfrak{V} satisfies:

$$\tilde{h}(x) = [1 - \exp(-\xi)] \sum_{t=0}^{\infty} \exp(-t\xi) \mathbb{E} e(X_t)^{-1} | X_0 = x$$

where the infinite sum converges in L^1 .

We conclude this subsection by making some comparisons between assumptions and the fixed points of the operators \mathfrak{U} and \mathfrak{V} as upper or lower bounds. A consequence of Jensen's inequality is that Assumption 4.1 implies Assumption 4.2 when $0 < \alpha < 1$ and conversely for $\alpha \geq 1$. For $\alpha < 0$, the assumptions are not comparable. For all three cases, we can apply Jensen's equality to rank fixed points of the $\tilde{\cdot}$ operators:

$$\begin{aligned} \tilde{h}(x)^{\frac{1}{\alpha}} &\leq \tilde{g}(x) & \alpha < 0 \\ \tilde{h}(x)^{\frac{1}{\alpha}} &\leq \tilde{g}(x) & 0 < \alpha < 1 \\ \tilde{h}(x)^{\frac{1}{\alpha}} &\geq \tilde{g}(x) & \alpha \geq 1. \end{aligned}$$

4.3 Candidate fixed points for \mathfrak{U} and \mathfrak{V}

We now use monotonicity to construct candidate fixed points for \mathfrak{U} and \mathfrak{V} . We consider three cases associated with three different intervals for α .

4.3.1 $\alpha < 0$

When Assumption 4.2 is satisfied,

$$\mathfrak{V}\tilde{h}(x) \leq \mathfrak{V}\tilde{h}(x) = \tilde{h}(x).$$

Thus

$$\mathfrak{V}^2\tilde{h}(x) \leq \mathfrak{V}\tilde{h}(x) \leq \tilde{h}(x),$$

and more generally $\bigcap_{j=0}^{\infty} \mathfrak{V}^j\tilde{h}$ is a decreasing sequence of functions. This sequence converges pointwise to a limit function \hat{h} . We will establish below that this limit solves the fixed-point equation:

$$\hat{h} = \mathfrak{V}\hat{h}$$

When Assumption 4.1 is satisfied, we construct a fixed point \hat{g} using the decreasing

sequence $\{\mathfrak{U}^j \tilde{g}\}^{\infty}$. In this case the limit solves:

$$\hat{g} = \mathfrak{U} \hat{g}$$

Since $\tilde{h}^{\frac{1}{\alpha}} \leq \tilde{g}$,

$$\mathfrak{V}^j \tilde{h}^{\frac{1}{\alpha}} = \mathfrak{U}^j \tilde{h}^{\frac{1}{\alpha}} \leq \mathfrak{U}^j \tilde{g}.$$

Taking limits as j tends to infinity,

$$\hat{h}^{\frac{1}{\alpha}}(x) \leq \hat{g}(x)$$

when Assumptions 4.1 and 4.2 are both satisfied.

4.3.2 $0 < \alpha < 1$

In this case we impose the more restrictive Assumption 4.1 and use \mathfrak{U} to construct a fixed point. Notice that

$$\mathfrak{V} \tilde{h} \leq \mathfrak{U} \tilde{h}^{\frac{1}{\alpha}} \leq \mathfrak{U} \tilde{g}^{\alpha} = \tilde{g}^{\alpha}$$

Applying \mathfrak{V} to both sides,

$$\mathfrak{V}^2 \tilde{h} \leq \mathfrak{V} \mathfrak{U} \tilde{h}^{\frac{1}{\alpha}} \leq \mathfrak{U}^2 \tilde{h}^{\frac{1}{\alpha}} \leq \mathfrak{U}^2 \tilde{g}^{\alpha} = \tilde{g}^{\alpha}.$$

Repeating this argument, we see that

$$\mathfrak{V}^j \tilde{h} \leq \tilde{g}^{\alpha}.$$

Also

$$\mathfrak{V} \tilde{h} \geq \mathfrak{V} \tilde{h} = \tilde{h}$$

Thus

$$\mathfrak{V}^2 \tilde{h}(x) \geq \mathfrak{V} \tilde{h} \geq \tilde{h}.$$

Consequently $\{\mathfrak{V}^j \tilde{h} : j = 1, 2, \dots\}$ is an increasing sequence of functions with a finite upper bound. This sequence converges pointwise to a limit function \hat{h} .

Since $\tilde{h}^{\frac{1}{\alpha}} \leq \tilde{g}$,

$$\mathfrak{V}^j \tilde{h}^{\frac{1}{\alpha}} = \mathfrak{U}^j \tilde{h}^{\frac{1}{\alpha}} \leq \mathfrak{U}^j \tilde{g}.$$

Taking limits as j tends to infinity

$$\hat{h}(x)^{\frac{1}{\alpha}} \leq \hat{g}(x).$$

4.3.3 $\alpha \geq 1$

In this case we impose the more restrictive Assumption 4.2 and use \mathfrak{V} to construct a decreasing sequence that is bounded from below by a strictly positive function. Hence the resulting sequence converges pointwise to a positive function \hat{g} . We use \mathfrak{U} to construct an increasing sequence that is bounded from above by a positive function. This sequence also converges. Finally, $\hat{h}^{\frac{1}{\alpha}} \geq \hat{g}$.

4.4 Extending the domain of convergence

We constructed fixed points by iterating operators starting from a specific function, say \tilde{g} , and converging to a limit point, say \hat{g} where $\hat{g} \leq \tilde{g}$. Consider a function g such that $\hat{g} \leq g \leq \tilde{g}$. Then

$$\hat{g} = \mathfrak{U}^j \hat{g} \leq \mathfrak{U}^j g \leq \mathfrak{U}^j \tilde{g}.$$

Since $\{\mathfrak{U}^j \tilde{g} : j = 1, 2, \dots\}$ converges to \hat{g} , $\{\mathfrak{U}^j g : j = 1, 2, \dots\}$ also converges to \hat{g} . This argument applies to all of the cases we have studied. At least in this specific sense, the candidate fixed points are “stable”.

5 Main result

We are now ready to state and prove our result on the existence of recursive utilities in a Markov framework. This proposition collects intermediate results from Section 4 and shows that the constructed fixed points are actually fixed points and that the fixed points coincide for $\alpha > 0$.

Proposition 5.1. *Suppose a) (X, Y) is a Markov process satisfying Assumption 2.1 holds; b) e is a solution to the Perron-Frobenius equation (9) satisfying Assumption 3.1 with $\exp(\eta)$ the associated eigenvalue; and c) the subjective rate of discount satisfies $\delta > \frac{\eta}{\alpha}$ (Assumption 3.2). Then for alternative ranges of α we have the following results.*

- i) If $\alpha < 0$, $\hat{h}^{\frac{1}{\alpha}}$ is a fixed point of \mathfrak{U} provided that Assumption 4.2 is satisfied, and \hat{g} is a fixed point of \mathfrak{U} provided that Assumption 4.1 is satisfied. When both assumptions are*

satisfied, $\hat{h}^{\frac{1}{\alpha}} \leq \hat{g}$.

ii) If $0 < \alpha < 1$, $\hat{h}^{\frac{1}{\alpha}} = \hat{g}$ is a fixed point of \mathfrak{U} provided that Assumption 4.2 is satisfied.

iii) If $\alpha \geq 1$, $\hat{h}^{\frac{1}{\alpha}} = \hat{g}$ is a fixed point of \mathfrak{U} provided that Assumption 4.1 is satisfied. Moreover, \hat{g} is the unique fixed point with a finite α moment under the $\tilde{\nu}$ stationary distribution

While the proposition features \mathfrak{U} , fixed points of \mathfrak{V} are constructed by raising the fixed points of \mathfrak{U} to the power α . Solutions for $\frac{V_t}{C_t}$ are given by multiplying fixed points of \mathfrak{U} by the eigenfunction $e^{\frac{1}{\alpha}}$ and raising the product to the power $\frac{1}{1-\rho}$.

We prove this proposition in the next two subsections. We first show that the limits we constructed in Section 4 are actually fixed points.

5.1 Existence of fixed points

To prove existence, we again treat three cases separately depending on the magnitude of α .

5.1.1 $\alpha < 0$

If Assumption 4.2 holds then $\tilde{h} \in L_+^1$ and $\mathfrak{V}^j \tilde{h}$ is a dominated sequence of L_+^1 functions converging pointwise to \hat{h} . The Dominated Convergence theorem guarantees that,

$$\lim_{j \rightarrow \infty} E[\mathfrak{V}^j \tilde{h}(X_{t+1} | X_t = x)] = E[\hat{h}(X_{t+1}) | X_t = x].$$

with $\tilde{\nu}$ measure one. Hence \hat{h} is a fixed point of \mathfrak{V} .

If Assumption 4.1 holds. Then, as we showed above, $\tilde{g} \in L_+^1$ and \mathfrak{U} maps L_+^1 into L_+^1 . Since $[1 - \exp(-\xi)]e(x)^{-\frac{1}{\alpha}} \leq \mathfrak{U}^j \tilde{g} \leq \tilde{g}(x)$, where the first inequality follows from bound (10), the dominated convergence theorem assures us that $\hat{g} \in L_+^1$ and is the strictly positive (with probability one) L^1 limit of $\mathfrak{U}^j \tilde{g}$. From inequality (12) it follows that for each j , $E[\mathfrak{U}^j \tilde{g}(X_{t+1}) | X_t = x] < \infty$. Let

$$A = \{x : \sup_j E[\mathfrak{U}^j \tilde{g}(X_{t+1}) | X_t = x] < \infty\}$$

Since $\alpha < 0$,

$$(\mathfrak{U}^{j+1} \tilde{g})^\alpha \geq (\mathfrak{U}^j \tilde{g})^\alpha.$$

Beppo Levi's Monotone Convergence theorem thus implies that for $x \in A$

$$\mathbb{E} \left[\mathfrak{U}^j \tilde{g}(X_{t+1}) \mid X_t = x \right] \rightarrow \mathbb{E}([\hat{g}(X_{t+1})]^\alpha \mid X_t = x),$$

and as a consequence $\bar{g}(x) = \lim_{j \rightarrow \infty} \mathfrak{U}^{j+1} \tilde{g}(x) = \mathfrak{U} \hat{g}(x)$, for $x \in A$. If $x \notin A$ then, $\mathbb{E} \left[\mathfrak{U}^j \tilde{g}(X_{t+1}) \mid X_t = x \right] \rightarrow \infty$ and,

$$\hat{g}(x) = [1 - \exp(-\xi)]e(x)^{-\frac{1}{\alpha}} \leq \mathfrak{U}g(x),$$

for any $g \in L_+^1$. Since $\hat{g} \leq \mathfrak{U}^j \tilde{g}$ and \mathfrak{U} is monotone, $\mathfrak{U} \hat{g} \leq \hat{g}$, and thus for $x \notin A$, we also have $\mathfrak{U} \hat{g}(x) = \hat{g}(x)$.

5.1.2 $0 < \alpha < 1$

If Assumption 4.1 holds, $\tilde{g} \in L_+^1$ and $\mathfrak{V}^j \tilde{h}$ is a sequence of L_+^1 functions dominated by $g^\alpha \in L_+^1$. The remainder of the proof is as above.

5.1.3 $\alpha > 1$

If Assumption 4.2 holds, the proof in 5.1.1 applies.

We next show that when $\alpha > 0$ the constructed fixed points are actually the same.

5.2 Coincidence of fixed points when $\alpha > 0$

Consistent with our prior analysis, we treat separately the the case in which $0 < \alpha < 1$ and $\alpha \geq 1$.

5.2.1 $0 < \alpha < 1$

Consider the function:

$$\psi(r) = s + \exp(-\xi)r^{\frac{1}{\alpha}}$$

for $r > 0$ and $s > 0$. The derivative of this function is:

$$\psi'(r) = \exp(-\xi) \frac{r^{\frac{1}{\alpha} - 1}}{s + \exp(-\xi)r^{\frac{1}{\alpha}}} = \exp(-\xi\alpha) \frac{\exp(-\xi)r^{\frac{1}{\alpha} - 1}}{s + \exp(-\xi)r^{\frac{1}{\alpha}}}.$$

This derivative is increasing in r and hence ψ is convex in r . Consequently for each fixed x , \mathfrak{V} is a convex function of $h > 0$. A subgradient for this convex function at h_1 is the linear map that maps a $k \in L_+^1$ into:

$$d(x) \mathfrak{E} [k(X_{t+1}) | X_t = x]$$

where

$$d(x) = \frac{2}{4} \frac{\exp(-\xi) \mathfrak{E} [h_1(X_{t+1}) | X_t = x]^{\frac{1}{\alpha}}}{[1 - \exp(-\xi)]e(x)^{\frac{-1}{\alpha}} + \exp(-\xi) \mathfrak{E} [h_1(X_{t+1}) | X_t = x]^{\frac{1}{\alpha}}} \frac{3_{1-\alpha}}{5} < 1$$

Thus if $h_1 \geq h_2$ are non-negative fixed points of \mathfrak{V} ,

$$h_2(x) - h_1(x) = \mathfrak{V}h_2(x) - \mathfrak{V}h_1(x) \geq d(x) \mathfrak{E} [(h_2 - h_1)(X_{t+1}) | X_t = x].$$

Integrate both sides with respect to the $\tilde{\tau}$ stationary distribution. By the Law of Iterated Expectations,

$$\mathfrak{E} [h_2(X_{t+1}) - h_1(X_{t+1})] \geq \mathfrak{E} (d(X_t) [(h_2 - h_1)(X_{t+1}) | X_t = x])$$

Since $0 < d(x) < 1$, $h_2 \leq h_1$, h_1 and h_2 coincide in a set with $\tilde{\tau}$ measure one. In particular $\hat{g}^\alpha = \hat{h}$.

5.2.2 $\alpha \geq 1$

We view $\mathfrak{E} [g(X_{t+1})^\alpha | X_t = x]^{\frac{1}{\alpha}}$ as a conditional norm. As a consequence, if $g_1 \geq 0$ and $g_2 \geq 0$ are fixed points of \mathfrak{U}

$$\begin{aligned} |g_1(x) - g_2(x)| &= \mathfrak{U}g_1(x) - \mathfrak{U}g_2(x) \\ &= \exp(-\xi) \mathfrak{E} [g_1(X_{t+1})^\alpha | X_t = x]^{\frac{1}{\alpha}} - \mathfrak{E} [g_2(X_{t+1})^\alpha | X_t = x]^{\frac{1}{\alpha}} \\ &\leq \exp(-\xi) \mathfrak{E} [|g_1(X_{t+1}) - g_2(X_{t+1})|^\alpha | X_t = x]^{\frac{1}{\alpha}} \end{aligned}$$

where the last relation follows from the (reverse) Triangle Inequality. Next raise both sides to the power α and then integrate with respect to the $\tilde{\tau}$ stationary distribution. By the

Law of Iterated Expectations, we find that

$$E [|g_1(X_{t+1}) - g_2(X_{t+1})|^\alpha] \leq \exp(-\xi) E [|g_1(X_{t+1}) - g_2(X_{t+1})|^\alpha].$$

provided that g_1 and g_2 have finite α moments under the $\tilde{\cdot}$ stationary distribution. Thus g_1 and g_2 must be equal with $\tilde{\cdot}$ probability one.

Since

$$\hat{g}^\alpha \leq \hat{h} \leq \tilde{h},$$

under Assumption 4.2 \hat{g} and \hat{h} ¹

recursion:

$$\mathcal{D}g(x) = \frac{\eta \exp(-\delta)}{1 - \gamma} - [1 - \exp(-\delta)] \log e(x) + \frac{\exp(-\delta)}{1 - \gamma} \mathbf{E}(\exp[(1 - \gamma)g(X_{t+1})] | X_t = x)$$

where we no longer restrict g to be positive. This recursion is a special case of the so-called “risk sensitive recursion” studied by Jacobson (1973) and Whittle (1990) where discounting is included in the manner suggested by Hansen and Sargent (1995). Let

$$\mathcal{E}g(x) = \frac{\eta \exp(-\delta)}{1 - \gamma} - [1 - \exp(-\delta)] \log e(x) + \exp(-\delta) \mathbf{E}[g(X_{t+1}) | X_t = x].$$

Then

$$\mathcal{D}g \leq \mathcal{E}g$$

and \mathcal{E} has a fixed point \tilde{g} provided that $\mathbf{E}[\log e(X_t)]$ is finite. We may use our previous arguments to show that $\{\mathcal{D}^j \tilde{g} : j = 1, 2, \dots\}$ is a decreasing sequence, but we do not have an obvious lower bound on these iterations. When they converge to a finite valued function \hat{g} , this function is a fixed point of \mathcal{D} .

Our analysis takes as given the consumption dynamics in contrast to stochastic growth economies such as those studied by Brock and Mirman (1972). The change of probability measure we use is determined by the multiplicative martingale component for consumption raised to a power as discussed in Hansen and Scheinkman (2009) and Hansen (2011). Some stochastic growth economies with production have a balanced growth path relative to some stochastically growing technology. In such economies, the value of η and the change of measure may be deduced prior to solving the model. In particular we may check parameter the restriction:

$$\delta > \frac{1 - \rho}{1 - \gamma} \eta$$

by solving for η using the exogenously specified technology and the balanced-growth restriction. This restriction on δ may be viewed as an extension of Kocherlakota (1990)’s analysis of subjective discount rates in stochastic growth economies for models with power utility preferences ($\gamma = \rho$). The eigenfunction e , which is also restricted in our analysis, will depend on a conjectured equilibrium solution for consumption, however.

7 Relation to large deviations

Donsker and Varadhan (1975, 1976) and many others use principal eigenvalue problems as a device for computing large deviation bounds. While their analysis allows for the construction of large deviation bounds for a large class of events, we consider bounding a rather simple set of tail events.

Following the work of Stutzer (2003), we explore the probabilities that consumption growth will be below some growth threshold at a given date.⁶ Consider the following threshold probability:

$$\mathcal{Pr} \{ \log C_t - \log C_0 \leq -rt | X_0 = x \} = \mathcal{Pr} \{ -\log C_t + \log C_0 - rt \geq 0 | X_0 = x \} \quad (14)$$

This probability is the “value at risk” that the growth rate of consumption will be less than $-r$. As we will eventually make the time horizon t tend to infinity, adding a constant to the threshold in (14) will be inconsequential. This computation is similar to but distinct from calculations for a class of ruin problems initiated by Cramer and Lundberg. See Nyrhinen (1999) for a more refined use than what we describe here of large deviation theory to compute asymptotic ruin probabilities.

To bound the probability in (14), we follow the usual approach to large deviations by constructing a family of functions that dominate the indicator function

$$x^\theta \geq \mathbf{1}_{\{\log x \geq 0\}}$$

for any $\theta \geq 0$. An implication of this domination is:

$$\exp(-\theta r t) E \left[\frac{C_t}{C_0} \right] | X_0 = x \geq \mathcal{Pr} \{ -\log C_t + \log C_0 - rt \geq 0 | X_0 = x \},$$

or in logarithms,

$$-\theta r + \frac{1}{t} \log E \left[\frac{C_t}{C_0} \right] | X_0 = x \geq \frac{1}{t} \log \mathcal{Pr} \{ \log C_t - \log C_0 + rt \leq 0 | X_0 = x \}$$

where we scaled by t . This bound holds for all $\theta > 0$, which leads us to minimize the

⁶Stutzer (2003) actually investigates the behavior of portfolios over long investment horizons while we look at consumption growth.

left-hand side with respect to θ . We will study the limiting result as the time horizon becomes large. The optimized θ depends on the growth rate r used in constructing the threshold of interest. We will link the choice of θ to the preference parameter $\gamma - 1$, and, as a consequence, the inverse problem will be of interest to us. Given θ , for what value of the growth rate threshold r will this θ be the best choice for constructing a large-deviation bound?

The large t approximation to the left-hand side is:

$$-\theta r + \eta(\theta). \tag{15}$$

where $\eta(\theta)$ is the Perron-Frobenius eigenvalue obtained by solving:

$$E(\exp[-\theta\kappa(X_{t+1}, Y_{t+1}, X_t)]e(X_{t+1})|X_t = x) = \exp[\eta(\theta)]e(x).$$

To construct the best possible asymptotic bound we minimize (15) with respect to $\theta \geq 0$, or equivalently:

$$\xi(r) = \sup_{\theta \geq 0} r\theta - \eta(\theta),$$

which is a Legendre transform. The function η can be shown to be convex in θ as is the Legendre transform ξ . With this construction, the decay rate in the probabilities for threshold r is $\xi(r)$. The first-order conditions are:

$$r = \eta'(\theta) = -\mathbb{E}[\kappa(X_{t+1}, Y_{t+1}, X_t)]$$

provided that η is differentiable where the distorted distribution is evaluated at the optimized value of θ . This same change in probability distribution is commonly used to verify that the upper bound just computed is also the best possible bound.

So far we have taken r to be specified and we solve for θ . We now consider the inverse problem by computing a threshold r that solves the optimization problem for a given θ . We solve this inverse problem to build a connection to our earlier analysis of intertemporal utility functions. To make this link, suppose that $\gamma > 1$ and let $\theta = \gamma - 1$. For each such value of γ , we compute a threshold for which the the power specification for terminal consumption gives the best probability bound.

We illustrate these calculations using a specification from Hansen et al. (2007) of a “long-run risk” model for consumption dynamics featured by Bansal and Yaron (2004).

Bansal and Yaron (2004) use historical data from the United States to motivate their model including the choice of parameters. Their model includes predictability in both conditional means and in conditional volatility. We use the continuous-time specification from Hansen et al. (2007) because the continuous-time specification of stochastic volatility is more tractable:

$$\begin{aligned}
 dX_t^{[1]} &= -.021X_t^{[1]}dt + X_t^{[2]} \begin{bmatrix} .00031 & -.00015 & 0 \end{bmatrix} dW_t, \\
 dX_t^{[2]} &= -.013(X_t^{[2]} - 1)dt + \begin{bmatrix} 0 & 0 & -.038 \end{bmatrix} dW_t, \\
 d\log C_t &= .0015dt + X_t^{[1]}dt + X_t^{[2]} \begin{bmatrix} .0034 & 0.007 & 0 \end{bmatrix} dW_t,
 \end{aligned}$$

where W is a trivariate standard Brownian motion. The unit of time in this specification is one month. The first component of the state vector is the state dependent component to the conditional growth rate, and the second component is a volatility state.⁷ Both the growth state and the volatility state are persistent. The average (in logarithms) growth rate in consumption in this example is .0015.

Our analysis assumes a discrete-time model. A continuous-time Markov process X observed at say interval points in time remains a Markov process in discrete time. Since $\log C_{t+1} - \log C_t$ is constructed via integration, it is not an exact function of X_{t+1} and X_t . To apply our analysis, we define $Y_{t+1} = \log C_{t+1} - \log C_t$. Given the continuous-time Markov specification, the joint distribution of $\log C_{t+1} - \log C_t$ and X_{t+1} conditioned on past information only depends on the current Markov state X_t as required by Assumption 2.1. We use the implied discrete-time specification to construct preferences and analyze implications. In light of this construction of a discrete time process from a continuous time starting point, we can exploit the continuous-time quasi analytical formulas given by Hansen (2011) for $\eta(\theta)$ as an important input into our calculations.⁸

⁷We follow Hansen (2011) in constructing this example. Hansen configures the shocks so that the first one is the “permanent shock” identified using standard time series methods and normalized to have a unit standard deviation. The second shock is a so-called temporary shock, which by construction is uncorrelated with the first shock.

⁸The formulas in Hansen (2011) for the continuous-time change of measure also may be used to characterize the discrete-time process under the change of measure with an analogous construction from a continuous-time process.

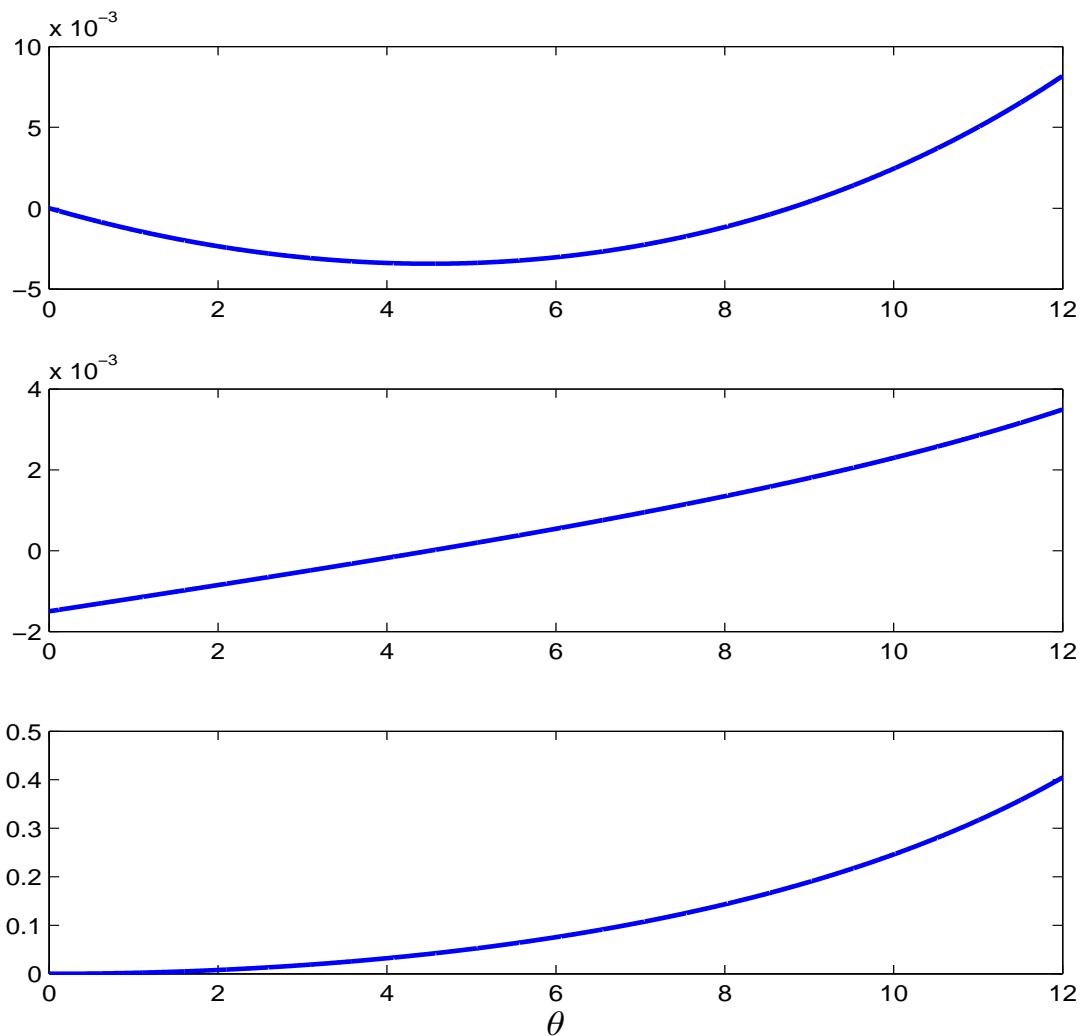


Figure 1: The top panel plots the logarithm of the eigenvalue η as a function of θ . The middle panel plots the implied threshold r for each value of θ . The bottom panel gives the implied decay rate in the probabilities for each value of θ . The decay rates are annualized.

We explore the consequences of changes in θ and implicitly for γ in Figure 1. This figure consists of three panels. The top panel gives the logarithm, η , as a function of $\theta = \gamma - 1$. As we argued previously, we expect η to be negative for at least modest values of θ . For larger values of θ stochastic volatility in the logarithm of the growth rate in consumption becomes sufficiently important that the expected growth rate in $(C)^{-\theta}$ becomes positive. For our example, this occurs for $\theta = \gamma - 1 > 8.76$. The second panel depicts the threshold r for the

which the value of θ on the horizontal axis is optimal. This is computed as $r(\theta) = \eta'(\theta)$. The unconditional mean of $\log C_{t+1} - \log C_t$ is .0015, and this is equal to $-r(0)$. For instance if r is set to the mean of the unconditional distribution of the growth rate (in logarithms) of consumption, after adjusting for mean growth rate and scaling by $\frac{1}{\sqrt{T}}$ the process obeys a Central Limit Theorem. Thus we do not expect decay in threshold probabilities, consistent with the zero decay rate for $\theta = 0$ in the bottom panel. As we reduce the threshold, the probabilities decay at a geometric rate. Positive values of θ , imply larger values of r , which corresponds to movements to the left tail of the distribution of $\log C_t$. The bottom panel gives the implied decay rates in the probabilities of consumption over an horizon t exceeding the threshold $-r(\theta)t$. This decay rate increases in θ as does $r(\theta)$. For instance when $r(\theta) = -.00075$, the decay rate is .0104 per annum and when $r(\theta) = 0$, the decay rate is .0408 per annum. The zero threshold $r(\theta) = 0$ occurs when $\theta = 4.51$ or equivalently $\gamma = 5.51$.

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