Crises in Repo Markets with Adverse Selection

Harald Uhlig*
University of Chicago, CentER, NBER and CEPR

Preliminary Working Paper

First draft: February 19th, 2010
This revision: September 24, 2010

Abstract

How can the worsening of a small part of the loan market lead to a crash as well as a prolonged depression in secondary loan prices, bank equity prices, and lending activity? This paper seeks to answer this question. We present a model in which banks issue long-term loans and finance them with repurchase agreements ("repos") from short-term lenders in order to leverage up their equity. Banks differ in their leveraging ability. Aside from the primary loan market ("loan-generation"), there is a secondary market, where loans can be traded among banks and sold by short-term lenders. Loans differ in quality, but the quality is observable only to the bank currently holding the loan contract. Therefore, the secondary loan market suffers from adverse selection, leading to important repercussions in loan origination. At some random time, the loan pool unexpectedly becomes heterogeneous. This leads to an initial crisis where highly levered banks are forced to sell loans, flooding the market and leading to fire-sale prices. In a numerical example, the initial date-$t = 0$ "cash in the market" crisis is followed by a prolonged period of subdued secondary loan prices, bank equity prices, and lending activity, in which the low-leverage banks rather than the high-leverage banks dominate the market.

*Address: Department of Economics, University of Chicago, 1126 East 59th Street, Chicago, IL 60637, U.S.A, email: huhlig@uchicago.edu. This research has been supported by the NSF grant SES-0922550. I am very grateful to Andrea Eisfeldt and Brett Green for fruitful discussions and initial exchanges of ideas on this topic, which inspired this paper.

1 Introduction

How did a modest increase in subprime US mortgage default rates turn into a world-wide financial crisis? Why did prices for collateralized debt assets decline so sharply, and, relatedly, why did the lending market and resale market for these securities dry up? How did the breakdown in these markets lead to the near collapse of the banking sector and to a significant decline in real loan activity?

This paper constructs a model which directly addresses these questions and investigates the impact of policies aimed to stabilize debt markets. In recent years, short term leverage has played a major role in the financing of debt securities. Several authors have documented the rise and subsequent fall in the quantity of debt securities financed in the repo market over the period before and after the financial crisis which began in the fall of 2008. They have also documented the sharp increase in risky debt yields, and the related spike in haircuts (collateral ratios) associated with borrowing against risky debt securities.[1] Our model is aimed at understanding the interaction between asset liquidity, bank solvency, and the ability to finance assets with leverage, when the secondary market for loans is tainted by adverse selection.

Banks and only banks can issue loans. Loans are long-term contracts. Banks are assumed to finance them with short-term repurchase agreements ("repos") with short-term lenders, leveraging up their equity. Aside from the primary loan market ("loan-generation"), there is a secondary market, where loans can be traded among banks and sold by short-term lenders. Loans may differ in quality, but the quality is observable only to to bank currently holding the loan contract. We assume that, starting at some period t=0, the loan pool unexpectedly becomes heterogeneous. This leads to rich implications for the ensuing dynamics. We show how this leads to an initial crash as well as a prolonged depression in secondary loan prices, bank equity prices, and lending activity.

Our model features a continuum of banks, who differ in their technologies which enable them to take on leverage. This heterogeneity seems important, and relevant, since much of the leverage of financial institutions relies on the "shadow" banking system, and on off balance sheet leverage that only sophisticated institutions can take advantage of. Consistent with our model, some very well known banks with prestigious reputations turned out to be the most highly levered and fragile institutions. For example, Lehman Brothers had several ways of increasing its off balance sheet leverage. These included using repo 105 transactions, and using a lender in which it had invested, Hudson Castle, to move risky assets off balance sheet via short term lending agreements.[2] These types of leverage technologies are likely

[1] See for example, Krishnamurthy (2010) and Gorton (2009).

[2] Repo 105 transactions were used by Lehman Brothers to move securities off their balance sheet at the end of the quarter with the intention of buying them back days later. Hudson Castle, whose largest

to be more costly for smaller, or less sophisticated banks. The difference in the ability to leverage in our model may also proxy for differences across institutional types, for example hedge funds can lever up more than commercial banks. However we maintain that there is important variation within institutional classes as well.[3] One interesting result of the heterogeneity that we assume is that the banks which have a better leveraging technology will seem "better" before the crisis. These banks will be more profitable, and they will grow more quickly. On average, the riskier banks will be larger, and this will make the financial system fragile. If a shock arrives which disrupts banks' ability to leverage, the banks that previously enjoyed a prestigious reputation and a large capacity for leverage will actually be the ones who will suffer the most. Thus, the previously "good" banks turn out to be "bad".[4]

2 The Model

Time is discrete, $t = \ldots, -2, -1, 0, 1, 2, \ldots$. There are two aggregate regimes, which we shall call "sunny" and "rainy". We assume that the aggregate regime switches from "sunny" to "rainy" on date $t = 0$, but that this was a completely unforeseen event. Alternatively, assume a Markov process for the regimes, where there is no exit from the "rainy" regime, and where the transition probability from the "sunny" to the "rainy" regime is tiny: we shall assume that it is too tiny to matter to enter considerations in the "sunny" regime. Time then counts the dates from the regime switch onwards, with $t = 0$ the first date of the "rainy" regime.

There is one type of good per period. There is a representative agent with expected linear utility in consumption of this good, discounting the future at rate $R \geq 1$. We assume

shareholder was Lehman, was a counterparty in several billion dollar transactions with similar balance sheet effects. Hudson Castle would lend cash to Lehman against its debt securities and Lehman would subsequently repurchase the securities and repay the loans. Acharya et al that the endowment is zero every period, but that consumption is allowed to be negative. Alternatively, assume that there is a sufficiently large endowment every period.

There are two types $\theta \in \{g, b\}$ of final technologies to transform resources across time. These technologies can be used at any date t. The technologies are linear, requiring the input of positive amounts of the good at date t. For every period afterwards, there is a constant probability $0 < 1 - \kappa \leq 1$ of termination[5]. The average lifetime until termination is therefore $1/(1-\kappa)$. In every period until (and including the period of) termination, the technologies pay some amount d_θ, $\theta \in \{s, g, b\}$ per unit invested. In the "sunny" regime, both technologies pay the same amount d_s. In the "rainy" regime, the amounts change to d_g for the g-technology and $d_b < d_g$ for the b-technology.

We shall call these technologies "loans". Moreover, we shall refer to the g-technology as a "good" loan and the b-technology as a "bad" loan. Heuristically, one may wish to imagine that there is a large and elastic demand for loans by borrowers, willing to repay d_θ for all periods until termination after receiving the initial loan of one unit. The amount varies in the rainy regime as some of the these borrowers become delinquent and therefore end up paying less.

It takes a special intermediary to operate a loan and collect its payments: we call that intermediary a bank. There is a continuum of intermediary types $\tau \in [\underline{\tau}, \bar{\tau}]$. Banks can invest available resources in loans. They can do so either by directly creating new loans ("primary market") or by purchasing loans on a secondary market. When creating new loans, an exogenously given share α of these new loans will be good, while $1 - \alpha$ will be bad. These loans are indistinguishable during the "sunny" regime. In the "rainy" regime, however, the bank learns the type of loans by observing the dividends. We assume that this information is entirely private to the bank holding the loan. After the loans have paid dividends in that period in period t, banks can buy and sell loans on the secondary market at a price p_t per unit of loan. Loan types cannot be distinguished on the secondary market and therefore they all fetch the same price.

We assume that there are short-term lenders, lending to banks at the discount rate R of the representative agent. These short-term loans take the form of one-period repurchase agreements ("repos"), with the terms of the agreement dependent on τ. More precisely, in order for a bank of type τ to extend a quantity L of loans, valued at market price p_t per unit, it can borrow up to $(1 - \tau)p_t L$ from the short-term lender, where τ can be interpreted as an (exogenously imposed) haircut or simply as the leveraging ability of the bank. For example, if a bank of type τ has one unit of resources available after its existing portfolio has paid dividends, repos have been repaid, and any part of the existing portfolio has been

[5]The termination assumption assures that a non-growing steady state features the financing of new loans.

sold then the bank can purchase up to

$$\lambda(\tau; p_t) = \frac{1}{\tau p_t} \tag{1}$$

units of loans on the secondary market, per borrowing $(1-\tau)p_t\lambda(\tau; p_t)$ from short-term lenders and investing $\tau p_t \lambda(\tau; p_t) = 1$ unit on its own. In the next period, the bank collects all the dividends on the loan, and purchases the loans back at the amount originally borrowed times the discount rate R. Alternatively, the bank may choose to create up to $\nu(\tau; p_t)$ new loans. The market value of these loans is $p_t\nu(\tau; p_t)$ and the bank can therefore borrow up to $(1-\tau)p_t\nu(\tau; p_t)$. A unit of new loans takes a unit of resources to create: therefore

$$\nu(\tau; p_t) = 1 + (1-\tau)p_t\nu(\tau; p_t) \tag{2}$$

or

$$\nu(\tau; p_t) = \frac{1}{1-(1-\tau)p_t} \tag{3}$$

In every period and every τ, new banks of type τ with total initial resources ("equity") $f(\tau)d\tau$ will be created, where $f(\cdot)$ is an exogenously given density with support $supp\{f\} = (\underline{\tau}, \bar{\tau})$. Therefore, the total new equity is $F = \int f(\tau)d\tau$. After collecting the dividends on its loan portfolio, any given bank will have to exit with the exogenously given probability $1-\gamma$ and will continue with probability $0 \leq \gamma < 1$. Exiting banks will sell their entire portfolio of non-terminated loans and repay the short-term lender up to the amount available (i.e., we assume limited liability). On the secondary market, the origin of the loans cannot be distinguished, i.e., a purchasing bank cannot see whether the loan was sold by an exiting bank or was sold by a continuing bank.

There is a timeline within each period. First, existing banks receive dividends on their loan portfolio and learn which parts of their non-terminated loan portfolio is good and which is bad. A fraction $1-\gamma$ of banks exit.[6] They sell their entire loan portfolio and repay the short-term lender, up to the amount available. All other banks choose which parts of the portfolio to sell, how many new loans to make, how many loans to purchase on the secondary market, how much to borrow from new short-term lenders and how much to pay as non-negative dividends to its owners. The bank must repay all existing repos, i.e. its short-term debt. If a bank runs out of funds to repay its repos (before paying dividends to its owners), it has to exit too ("declare bankruptcy"). Alternatively, one can assume that the bank continues with zero resources available.

New banks belong to the representative agent. Banks maximize the objective of its owner. They therefore take their portfolio and borrowing decision such that they maximize the expected discounted payoff to the representative agent.

[6]This assumption assures that banks will not grow unboundedly. An alternative interpretation is that a representative fraction of the banks loan portfolio must be sold or rebalanced for liquidity reasons

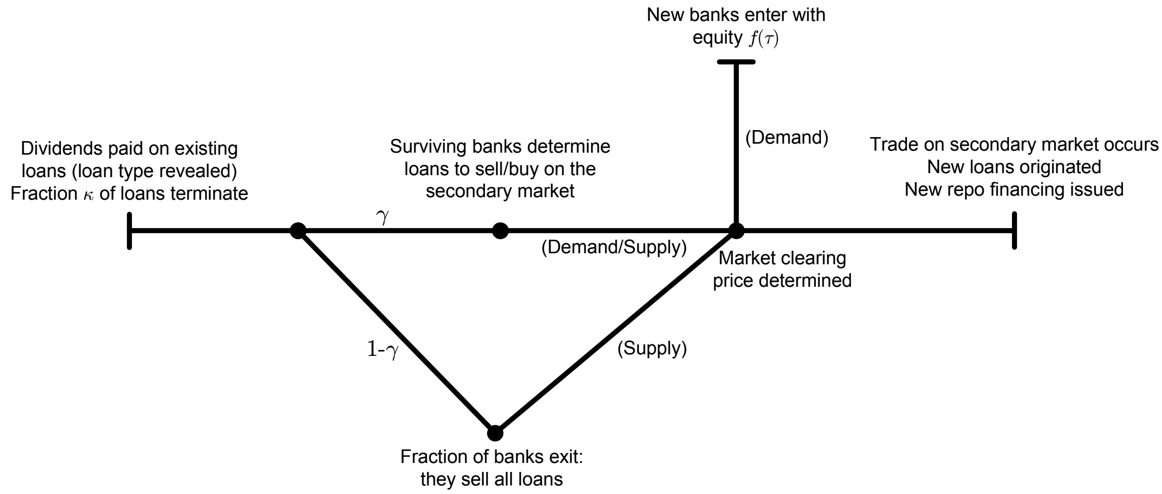


Figure 1: Timeline within each period

Let $L_g(\tau;t)$ and $L_b(\tau;t)$ be the entire portfolio of good loans and bad loans of all banks of type τ at the beginning of period t. More precisely, these should be read as densities $L_g(\tau;t)d\tau$ and $L_b(\tau;t)d\tau$ of the mass distribution of loans across bank-types τ. A stationary equilibrium for either regime is a price p on the secondary market for loans, as well selling, purchasing, new-loan issuance and borrowing decisions, such that the objective of the share-holding representative agent is maximized and such that $L_g(\tau;t)$ and $L_b(\tau;t)$ are independent of time.

For the non-stationary period, we shall assume that we emerge from the stationary "sunny" regime equilibrium at some random date $t = 0$. Let $L_g(\tau;0)$ and $L_b(\tau;0)$ be the entire portfolio of good loans and bad loans of all banks of type τ in the "sunny"-regime stationary equilibrium with $p = 1$ (see section 3). They are the given initial conditions for the "rainy" regime. An equilibrium is a sequence of prices $(p_t)_{t=0}^{\infty}$, a sequence of portfolios $L_g(\tau;t)$ and $L_b(\tau;t)$ of good loans and bad loans of all banks of type τ at the beginning of period t, as well as selling, purchasing, new-loan issuance and borrowing decisions at each $t \geq 0$, such that the objective of the share-holding representative agent is maximized.

It is useful to define

$$r = R - \kappa \tag{4}$$

which one may wish to think of as the required net return on the technologies from the perspective of the agent, given their random termination. The following assumption ensures that investing in the technologies is attractive both in the "sunny" as well as in the "rainy" regime.

Assumption 1.

$$\begin{aligned} r &< d_s &(5) \\ r &< \bar{d} = \alpha d_g + (1-\alpha) d_b \end{aligned}$$

A natural benchmark is to set

$$d_s = \bar{d} \tag{6}$$

i.e., to assume that average dividends per unit of loan do not change between the regimes, but that quality differences arise (or are noticed), as the economy transits from the "sunny" to the "rainy" regime.

3 The "sunny" regime.

We postulate that there is a stationary equilibrium with $p = 1$ for the "sunny" regime, where banks invest all newly earned equity in loans and only repay shareholders upon exiting. Furthermore, banks will leverage their equity maximally. We show this by constructing the equilibrium.

Note that

$$\nu(\tau; 1) = \lambda(\tau; 1)$$

Hence, a bank τ can issue as many new loans as it can purchase on the secondary market: with $p = 1$, banks are indifferent as to which market to be active on.

Let $L(\tau)$ be the total quantity of loans held by banks at the beginning of the period, and assume them to have been purchased with maximal leverage. The quantity $\gamma L(\tau)$ belongs to the surviving banks. Repaying all short-term loans and selling all loans leaves these banks with equity

$$e_s(\tau)\gamma L(\tau)$$

where

$$\begin{aligned} e_s(\tau) &= d_s + \kappa - R(1-\tau) \\ &= d_s - r + R\tau \end{aligned}$$

is the equity earned per unit of loan. Note that a loan can only be sold, if it has not been terminated: therefore, κ appears in this equation. Recall that $r = R - \kappa$. By assumption 1, equity generated is positive, and it therefore is optimal to keep the equity in the bank.

With the injection of fresh equity $f(\tau)$ from newly created banks, and leveraging the total bank-τ equity maximally for the purchase or creation of new loans $L'(\tau)$, one obtains

$$L'(\tau) = \frac{1}{\tau} \left(e_s(\tau)\gamma L(\tau) + f(\tau) \right)$$

Imposing stationarity $L(\tau) = L'(\tau)$ yields the solution

$$L(\tau) = \frac{f(\tau)}{\tau - \gamma e_s(\tau)} = \frac{f(\tau)}{\tau(1-\gamma R) - \gamma(d_s - r)} \tag{7}$$

In order for this solution to make economic sense, we restrict the parameters as follows.

Assumption 2.

$$\tau(1-\gamma R) - \gamma(d_s - r) > \epsilon \tag{8}$$

for all $\tau \in (\underline{\tau}, \bar{\tau})$ *and some small positive constant* $\epsilon > 0$.[7]

With this, the entire loan portfolio is

$$L = \int L(\tau) d\tau \leq \frac{F}{\epsilon} < \infty \tag{9}$$

The quantity $(1-\kappa)L$ of these loans terminates and is replaced by new loans, whereas κL is the quantity of loans which continue to the next period. Equation (8) can now be restated as the two conditions: $1 - \gamma R > 0$, and

$$\underline{\tau} > \frac{\epsilon + \gamma(d_s - r)}{1 - \gamma R} \tag{10}$$

We can calculate the value $q(\tau)$ of a unit of equity invested in a new bank. That unit of equity can be used to purchase $\lambda(\tau; 1) = 1/\tau$ unit of loans, fully levered. These in turn produce equity $e_s(\tau)/\tau$ next period: this will be paid out, in case the bank has to exit, or reinvested in loans otherwise. Thus,

$$q(\tau) = \frac{1 - \gamma + \gamma q(\tau)}{R} \frac{e_s(\tau)}{\tau}$$

which implies that

$$q(\tau) = \frac{(1-\gamma) e_s(\tau)}{R\tau - \gamma e_s(\tau)} \tag{11}$$

This formula compares well to its counterpart in equation (37), which holds in the adverse selection steady state.

Lemma 1. *With assumptions 1 and 2,* $\underline{\tau} > 0$*. Furthermore, for* $\tau \in (\underline{\tau}, \bar{\tau})$*,* $q(\tau)$ *is strictly decreasing in* τ *and satisfies*

$$q(\tau) < \frac{1-\gamma}{(R-1)\gamma} < \infty \tag{12}$$

If $R = 1$*, then*

$$q(\tau) \leq 1 + \frac{d_s - r}{\epsilon} < \infty$$

[7] This parameter restriction is sufficient, but not necessary: $\epsilon = 0$ in (8) would be enough to assure that $L(\tau)$ is a finite (or positive) solution in (7) and some slight additional assumption about integrability would then be enough to ensure a finite amount of loans L in (9).

Proof. It is easy to see that $\underline{\tau} > 0$. Rewrite equation (11) for $q(\tau)$ as

$$q(\tau) = \frac{1-\gamma}{\frac{R}{\frac{d_s - r}{\tau} + R} - \gamma} \tag{13}$$

This shows that $q(\tau)$ is decreasing in τ, provided $q(\tau)$ is finite. It therefore suffices to show (12) for $\tau = \underline{\tau}$. This is easy to verify. For $R > 1$, replace ϵ with 0 to obtain the upper bound. $\square$

It may be useful to write

$$q(\tau) = \frac{(1-\gamma)e_s(\tau)}{(1-\gamma)R\tau - \gamma(d_s - r)}$$

to compare the numerator here to the numerator for $L(\tau)$ in equation (7).

4 The "rainy" regime.

We analyze the "rainy" regime in three stages: the initial period $t = 0$, the transition for $t \geq 1$ and the steady state as $t \to \infty$. In subsection 4.1, we provide the forward evolution of the "rainy" regime for $t \geq 1$ onwards in the benchmark scenario, that banks never have to sell good loans and always find it in the interest of the shareholders to obtain maximal leverage. This requires restrictions on parameters and there may generally be other equilibria. In subsection 4.2, we discuss the stationary equilibrium for the "rainy" regime. We then turn to the special initial period $t = 0$ in subsection 4.3. That period provides initial conditions for the analysis of subsection 4.1, i.e. for the dynamics from $t \geq 1$ onwards.

4.1 The forward evolution for $t \geq 1$.

In this section, we derive the forward evolution of loan portfolios and prices in the "rainy" regime from $t = 1$ onwards for the benchmark scenario, in which the following high-level assumption holds:

Assumption 3. *Banks always find it in the interest of the share holders to obtain maximal leverage. Furthermore, for all $t > 0$, they never have to sell good loans or declare bankruptcy in order to repay short-term lenders.*

This assumption saves considerably on notation and on examining special cases and corners. However, it is obviously desirable to state more primitive conditions, deriving the assumption above as a result. One may either do so by examining the calculated equilibrium ex post, or by seeking out appropriate parameter restrictions.

We now proceed to construct the equilibrium. Given some period $t \geq 1$, let $L_g(\tau;t)$ and $L_b(\tau;t)$ be the portfolio of good and bad loans for banks of type τ at the beginning of period t. We assume them to be fully levered. Likewise, assume that we are given the price p_{t-1} and the price p_t of the current period. We shall solve for the loan portfolios $L_g(\tau;t+1)$ and $L_b(\tau;t+1)$ of the next period, and, eventually, for the price path $(p_t)_{t\geq 1}$.

Define

$$L(\tau;t) = L_g(\tau;t) + L_b(\tau;t) \tag{14}$$

and let

$$\begin{aligned} L_{g,t} &= \int L_g(\tau;t)d\tau \\ L_{b,t} &= \int L_b(\tau;t)d\tau \\ L_t &= L_{g,t} + L_{b,t} = \int L(\tau;t)d\tau \end{aligned}$$

be the total portfolio in good and bad loans, and loans in total. Of these loans, the fraction κ is not terminated and may be traded on the secondary market. The equilibrium will have the feature that all bad loans are sold and that good loans are sold only by exiting banks. Therefore, the fraction of good loans in the secondary market is given by

$$\tilde{\alpha}_t = \frac{(1-\gamma)\kappa L_{g,t}}{(1-\gamma)\kappa L_t + \gamma\kappa L_{b,t}} \tag{15}$$

Note that loans are traded only among banks. Therefore the total loan portfolio L_t must split into a fraction α of good loans, $L_{g,t} = \alpha L_t$, and $1-\alpha$ of bad loans, $L_{b,t} = (1-\alpha)L_t$. With that, we obtain,

$$\tilde{\alpha}_t \equiv \tilde{\alpha} = \frac{(1-\gamma)\alpha}{1-\gamma\alpha} \tag{16}$$

i.e., we obtain a constant fraction of good loans in the secondary market. For the analysis of certain policy interventions, one may wish to analyze the case of a non-constant $\tilde{\alpha}_t$, though.

Introduce the following notation. Let $q_t(\tau)$ be the value for a unit of equity inside a bank of type τ at time t. Let $v_{\theta,t}(\tau)$ be the value to a bank-τ shareholder of a type $\theta \in \{g,b\}$ loan at the end of period t. Suppose a loan of type θ is sold and the short-term loan is repaid. This generates cash flow

$$e_\theta(\tau;p_{t-1},p_t) = d_\theta + \kappa p_t - Rp_{t-1}(1-\tau) \tag{17}$$

Alternatively, the bank could retain and re-leverage the loan instead of selling it. In this case, there is no κp_t from selling the loan, but $\kappa p_t(1-\tau)$ from leveraging the loan again, which can be used to repay the previous short-term financing $Rp_{t-1}(1-\tau)$. The net equity generation is

$$\begin{aligned} n_\theta(\tau;p_{t-1},p_t) &= d_\theta + (\kappa p_t - Rp_{t-1})(1-\tau) \\ &= e_\theta(\tau;p_{t-1},p_t) - p_t\kappa\tau \end{aligned}$$

With this, we can characterize the value of a good loan and a bad loan:

$$\begin{aligned} v_{\theta,t-1}(\tau) &= \frac{1-\gamma}{R}e_\theta(\tau;p_{t-1},p_t) + \frac{\gamma}{R}\left(\max\left\{e_\theta(\tau;p_{t-1},p_t)q_t(\tau), n_\theta(\tau;p_{t-1},p_t)q_t(\tau) + \kappa v_{\theta,t}(\tau)\right\}\right) \\ &= \frac{1-\gamma+\gamma q_t(\tau)}{R}e_\theta(\tau;p_{t-1},p_t) + \frac{\gamma\kappa}{R}\left(v_{\theta,t}(\tau) - \tau p_t q_t(\tau)\right)^+ \end{aligned} \tag{18}$$

where $(x)^+ = \max\{x,0\}$. Equation (18) highlights the relevant comparison as to whether the bank wants to sell a type-θ loan or hold and re-leverage it is $\tau p_t q_t(\tau)$ versus $v_{\theta,t}(\tau)$. The former represents the value of the additional equity the bank gains by selling the loan while the latter represents the value to the bank of the remaining loan (both are scaled by $1/\kappa$). It will be useful to define $\pi_{\theta,t}(\tau) \equiv v_{\theta,t} - \tau p_t q_t(\tau)$ as the *option value* of holding and re-leveraging the loan versus selling it. Of course, we will need the price path in order to determine the option value. Nevertheless, we have the following useful result.

Lemma 2. *For ho7.7011Tf 6.64TJ*

The term on the second line can be interpreted as an option value: the difference between the value $v_{g,t}(\tau)$ of hanging on to the fully levered good loan versus selling it at price p_t, recovering the equity portion τp_t and re-investing this equity at an instantaneous payoff $q_t(\tau)$, provided the loan has not terminated.

A bad loan will always be sold and the proceeds will be re-invested provided the bank does not exit. Therefore,

$$v_{b,t-1}(\tau) \quad = \quad \frac{1-\gamma+\gamma q_t(\tau)}{R} e_b(\tau; p_{t-1}, p_t)$$

With these two values, we can calculate the value $v_{t-1}(\tau)$ of a fully leveraged newly generated loan as well as the value $\tilde{v}_{t-1}(\tau)$ of a loan on the secondary market, where the qualities are unknown, and where one has to take into account the likelihood of obtaining a good or a bad loan. Define

$$\begin{aligned} e(\tau; p_{t-1}, p_t) &= \alpha e_g(\tau; p_{t-1}, p_t) + (1-\alpha) e_b(\tau; p_{t-1}, p_t) \\ \tilde{e}(\tau; p_{t-1}, p_t) &= \tilde{\alpha} e_g(\tau; p_{t-1}, p_t) + (1-\tilde{\alpha}) e_b(\tau; p_{t-1}, p_t) \end{aligned}$$

as the average gross cash flow for newly created loans or for loans on the secondary market, assuming the loans get sold. With that, the value of a new loan is

$$\begin{aligned} v_{t-1}(\tau) &= \alpha v_{g,t-1}(\tau) + (1-\alpha) v_{b,t-1}(\tau) \\ &= \frac{1-\gamma+\gamma q_t(\tau)}{R} e(\tau; p_{t-1}, p_t) \\ &\quad + \frac{\alpha\kappa\gamma}{R} \left(v_{g,t}(\tau) - p_t \tau q_t(\tau) \right) \end{aligned}$$

The value of a loan on the secondary market is

$$\begin{aligned} \tilde{v}_{t-1}(\tau) &= \tilde{\alpha} v_{g,t-1}(\tau) + (1-\tilde{\alpha}) v_{b,t-1}(\tau) \\ &= \frac{1-\gamma+\gamma q_t(\tau)}{R} \tilde{e}(\tau; p_{t-1}, p_t) \\ &\quad + \frac{\tilde{\alpha}\kappa\gamma}{R} \left(v_{g,t}(\tau) - p_t \tau q_t(\tau) \right) \end{aligned}$$

To calculate the value $q_t(\tau)$ of a unit of equity, consider now the following two possibilities, assuming maximal leverage. First, a bank can invest in entirely new loans, obtaining $\nu(\tau; p_t)$ new loan units, with a fraction α of good loans. Second, a bank can invest in the secondary market and obtain $\lambda(\tau; p_t)$ loans, with a fraction $\tilde{\alpha}$ of good loans. We obtain:

$$q_t(\tau) = \max\{\nu(\tau; p_t) v_t(\tau); \lambda(\tau; p_t) \tilde{v}_t(\tau)\} \tag{19}$$

Put differently: the bank chooses to invest in new loans, if

$$\nu(\tau; p_t) v_t(\tau) > \lambda(\tau; p_t) \tilde{v}_t(\tau) \tag{20}$$

it chooses to invest in loans on the secondary market, if

$$\nu(\tau;p_t)v_t(\tau) < \lambda(\tau;p_t)\tilde{v}_t(\tau) \tag{21}$$

and is indifferent, if

$$\nu(\tau;p_t)v_t(\tau) = \lambda(\tau;p_t)\tilde{v}_t(\tau) \tag{22}$$

Proposition 1. *For all $t \geq 1$, there exists $\tau_t^* \in (\underline{\tau}, \bar{\tau})$ such that:*

1. *Banks with $\tau > \tau_t^*$ are active only in the primary market: (20) holds $\forall\, \tau > \tau_t^*$.*

2. *Banks with $\tau < \tau_t^*$ are active only in the secondary market: (21) holds $\forall\, \tau < \tau_t^*$.*

Proof. TO BE COMPLETED. □

With this, we can now state the supply and the demand for secondary loans as a function of the current price p_t, holding everything at $t-1$ and earlier and everything at $t+1$ and later fixed.

Given Lemma 2, the supply is

$$\begin{aligned} S(p,t) &= (1-\gamma)\kappa L_t + \gamma\kappa L_{b,t} \\ &= (1-\gamma+\gamma(1-\alpha))\kappa L_t \\ &= (1-\gamma\alpha)\kappa L_t \end{aligned}$$

and does not depend on p_t (except via the periods $t-1$ or $t+1$).

To calculate demand, observe that all loans are offered on the secondary market, except for the fraction of good loans among the continuing banks. Therefore, let $X_t(\tau;p_{t-1},p_t)$ be the total equity of type-τ banks for reinvestment in loans, including the new equity infusion $f(\tau)$. It is given by

$$\begin{aligned} X_t(\tau;p_{t-1},p_t) &= \gamma L_{g,t}(\tau)(d_g + (\kappa p_t - R p_{t-1})(1-\tau)) \\ &\quad +\gamma L_{b,t}(\tau)(d_b + \kappa p_t - R p_{t-1}(1-\tau)) + f(\tau) \\ &= \gamma\left(L_{g,t}(\tau)(d_g - \tau\kappa p_t) + L_{b,t}(\tau)d_b + L_t(\tau)(\kappa p_t - R p_{t-1})(1-\tau)\right) + f(\tau) \end{aligned} \tag{23}$$

Market demand is given by

$$D(p_t) = \frac{1}{p_t}\int_{\tau \leq \tau_t^*} \frac{1}{\tau} X_t(\tau;p_{t-1},p_t)\,d\tau$$

The loan evolution is now given as follows. Pick a τ. The initial portfolio of good and bad loans is $L_{b,t}(\tau)$ and $L_{g,t}(\tau)$. Of these, a fraction κ remains without termination and can be traded on the secondary market. A fraction $1-\gamma$ of these will be sold, due to exit. All remaining bad loans will be sold.

Suppose that $\tau \leq \tau_t^*$: wlog, we shall include τ^* in the highly leveraged secondary market. In that case, all of X_t will be invested in secondary loans, levered up by $1/\tau$. A fraction $\tilde{\alpha}$ of these loans are of good quality. After trading in the secondary market, the new quantities of good and bad quality loans held by banks of type τ is

$$L_{g,t+1}(\tau) = \gamma\kappa L_{g,t}(\tau) + \frac{\tilde{\alpha}}{\tau p_t} X_t(\tau; p_{t-1}, p_t) \tag{24}$$

$$L_{b,t+1}(\tau) = \frac{1-\tilde{\alpha}}{\tau p_t} X_t(\tau; p_{t-1}, p_t) \tag{25}$$

Suppose that $\tau > \tau_t^*$. In that case, all of X_t will be invested in primary loans. We get

$$L_{g,t+1}(\tau) = \gamma\kappa L_{g,t}(\tau) + \frac{\alpha}{1-(1-\tau)p_t} X_t(\tau; p_{t-1}, p_t) \tag{26}$$

$$L_{b,t+1}(\tau) = \frac{1-\alpha}{1-(1-\tau)p_t} X_t(\tau; p_{t-1}, p_t) \tag{27}$$

It may be instructive to calculate the share $\alpha(\tau)$ of good loans held within the portfolio of type-τ banks. I.e., define

$$\alpha_t(\tau) = \frac{L_{g,t}(\tau)}{L_t(\tau)} \tag{28}$$

In general, one may not expect $\tilde{\alpha}_t(\tau)$ to be constant over time. Furthermore, one should typically expect $\tilde{\alpha}_t(\tau)$ to differ across various values of τ.

4.2 The stationary equilibrium of the "rainy" regime.

In the stationary equilibrium, all variables above are constant. In particular, there is a constant price p on the secondary market. Given p, one can obtain the critical type $\tau^*(p)$, which splits the banks into those trading on the secondary or on the primary market. Equations (24) through (27) constitute two linear systems of two equations in two unknowns $L_g(\tau; p)$ and $L_b(\tau; p)$: one system is for $\tau \leq \tau^*(p)$, whereas the other system is for $\tau > \tau^*(p)$. These systems can be solved. Given these solutions, one can then calculate demand and supply for secondary loans, and solve for p.

In the "rainy" steady state, assumption 3 has to hold, as the following proposition shows.

Proposition 2. *In the steady state of the "rainy" regime, banks always find it in the interest of the share holders to obtain maximal leverage. Furthermore, they never have to sell good loans or declare bankruptcy in order to repay short-term lenders.*

Proof. TO BE COMPLETED. Heuristically, there must be investment in new loans: otherwise, with the continuous exit of banks, demand for secondary loans will outstrip supply eventually at all costs below new loans. On the other hand, the secondary market must be in operation too, as exiting banks seek to sell their loans. As prices do not change, and leverage is calculated with respect to market value, it is optimal to achieve maximal leverage. □

We restate the equations characterizing the steady state equilibrium, eliminating the dependencies on variables which will be constant, except for the terms stating generated equity. The analysis parallels the dynamic analysis above: while it is useful to restate it in the same sequence, we shall abbreviate the explanation.

Define the equity generated from selling a loan of type x as

$$\begin{aligned} e_x(\tau; p, p) &= d_x + \kappa p - Rp(1-\tau) \\ &= d_x - rp + Rp\tau \end{aligned}$$

However, only bad loans will be sold by non-exiting banks, while good loans will be retained on non-exiting banks balance sheets, again at maximal leverage (Lemma 2).

The value of a good loan at maximal leverage is given implicitly by

$$v_g(\tau) = \frac{1-\gamma+\gamma q(\tau)}{R} e_g(\tau; p, p) + \frac{\kappa\gamma}{R} \left(v_g(\tau) - p\tau q(\tau)\right)$$

where, as before, the second term can be interpreted as an option value of holding the good loan (provided it has not terminated). Given $q(\tau)$, this can be solved to

$$v_g(\tau) = \frac{1-\gamma}{R-\kappa\gamma} e_g(\tau; p, p) + \frac{\gamma}{R-\kappa\gamma} \left(e_g(\tau; p, p) - \kappa p\tau\right) q(\tau) \tag{29}$$

A bad loan will always be sold and the proceeds will be reinvested. Therefore and assuming maximal leverage, its value is

$$v_b(\tau) = \frac{1-\gamma}{R} e_b(\tau; p, p) + \frac{\gamma}{R} e_b(\tau; p, p) q(\tau)$$

With that, the value of a new loan is

$$v(\tau) = \alpha v_g(\tau) + (1-\alpha) v_b(\tau)$$

The value of a loan on the secondary market is

$$\tilde{v}(\tau) = \tilde{\alpha} v_g(\tau) + (1-\tilde{\alpha}) v_b(\tau)$$

The steady state value $q(\tau)$ of a unit of equity is given by

$$q(\tau) = \max\{\nu(\tau; p) v(\tau); \lambda(\tau; p)\tilde{v}(\tau)\} \tag{30}$$

The bank only issues new loans, if

$$\nu(\tau; p) v(\tau) > \lambda(\tau; p)\tilde{v}(\tau) \tag{31}$$

It invests only in loans on the secondary market, if

$$\nu(\tau; p) v(\tau) < \lambda(\tau; p)\tilde{v}(\tau) \tag{32}$$

and is indifferent, if

$$\nu(\tau;p_t)v(\tau)=\lambda(\tau;p_t)\tilde{v}(\tau) \tag{33}$$

For these cases, one can now solve for the values explicitly. If (31) holds for bank τ, then $q(\tau)=v(\tau)/(1-(1-\tau)p)$. With the equations above, one now gets

$$v(\tau)=\frac{A(\tau,p)}{1-\frac{1}{1-(1-\tau)p}B(\tau,p)} \tag{34}$$

and the implied stock market value for always investing on the primary market is

$$q_{\text{prim}}(\tau)=\frac{1}{1-(1-\tau)p}v(\tau)=\frac{A(\tau,p)}{\tau p+1-p-B(\tau,p)} \tag{35}$$

where

$$\begin{aligned}
A(\tau,p) &= \alpha\frac{1-\gamma}{R-\kappa\gamma}e_g(\tau;p,p)+(1-\alpha)\frac{1-\gamma}{R}e_b(\tau;p,p)\\
B(\tau,p) &= \alpha\frac{\gamma}{R-\kappa\gamma}\left(e_g(\tau;p,p)-p\kappa\tau\right)+(1-\alpha)\frac{\gamma}{R}e_b(\tau;p,p)\\
&= \frac{\gamma}{1-\gamma}A(\tau,p)-\frac{\alpha\kappa\gamma}{R-\kappa\gamma}p\tau
\end{aligned}$$

Note that therefore

$$q_{\text{prim}}(\tau) = \frac{1}{\frac{1}{A(\tau,p)}\left(\tau p(1+\frac{\alpha\kappa\gamma}{R-\kappa\gamma})+1-p\right)-\frac{\gamma}{1-\gamma}}$$

which will be useful for further analysis.

Likewise, if (32), then $q(\tau)=\tilde{v}(\tau)/(\tau p)$. With the equations above, one now gets

$$\tilde{v}(\tau)=\frac{\tilde{A}(\tau,p)}{1-\frac{1}{\tau p}\tilde{B}(\tau,p)} \tag{36}$$

Similar to equation (11), the implied stock market value for always investing on the secondary market is

$$q_{\text{sec}}(\tau)=\frac{1}{\tau p}\tilde{v}(\tau)=\frac{\tilde{A}(\tau,p)}{\tau p-\tilde{B}(\tau,p)} \tag{37}$$

where

$$\begin{aligned}
\tilde{A}(\tau,p) &= \tilde{\alpha}\frac{1-\gamma}{R-\kappa\gamma}e_g(\tau;p,p)+(1-\tilde{\alpha})\frac{1-\gamma}{R}e_b(\tau;p,p)\\
\tilde{B}(\tau,p) &= \tilde{\alpha}\frac{\gamma}{R-\kappa\gamma}\left(e_g(\tau;p,p)-p\kappa\tau\right)+(1-\tilde{\alpha})\frac{\gamma}{R}e_b(\tau;p,p)\\
&= \frac{\gamma}{1-\gamma}\tilde{A}(\tau,p)-\frac{\tilde{\alpha}\kappa\gamma}{R-\kappa\gamma}p\tau
\end{aligned}$$

Note that this implies

$$q_{\text{sec}}(\tau) \quad = \quad \frac{1}{\frac{1}{\tilde{A}(\tau,p)}\tau p\left(1+\frac{\tilde{\alpha}\kappa\gamma}{R-\kappa\gamma}\right)-\frac{\gamma}{1-\gamma}}$$

which will be useful for further analysis. Numerically, one may now proceed to simply calculate $v(\tau)$ and $\tilde{v}(\tau)$ for all τ, and compare, in order to choose the best strategy.

Proposition 3. *Given p, there exists a $\tau^*(p)$, such that*

1. *Banks with $\tau > \tau^*(p)$ are active only in the primary market: $q_{prim}(\tau) > q_{sec}(\tau)$ $\forall\,\tau > \tau^*(p)$.*

2. *Banks with $\tau < \tau^*(p)$ are active only in the secondary market: $q_{prim}(\tau) < q_{sec}(\tau)$ $\forall\,\tau < \tau^*(p)$.*

Furthermore, $\tau^*(p) > \underline{\tau}$ and is decreasing in p.

Proof. TO BE COMPLETED. □

The equity of type-τ banks for reinvestment in loans is

$$X(\tau;p,p) \quad = \quad \gamma L_g(\tau)(d_g - rp(1-\tau)) + \gamma L_b(\tau)(d_b - rp + Rp\tau) + f(\tau) \tag{38}$$

For $\tau \leq \tau^*$, all of $X(\tau)$ will be invested in secondary loans, levered up by $1/\tau p$. A fraction $\tilde{\alpha}$ of these loans are of good quality. Therefore, the new volume of good and bad quality loans is

$$L_g(\tau) \quad = \quad \frac{\tilde{\alpha}}{\tau p}X(\tau;p,p) + \gamma\kappa L_g(\tau) \tag{39}$$

$$L_b(\tau) \quad = \quad \frac{1-\tilde{\alpha}}{\tau p}X(\tau;p,p) \tag{40}$$

With (38), this becomes a system of two equations in the two unknowns: $L_g(\tau)$ and $L_b(\tau)$, given p. As a first consequence, note

$$(1-\tilde{\alpha})(1-\kappa\gamma)L_g(\tau) = \tilde{\alpha}L_b(\tau)$$

or

$$L_g(\tau) = \frac{\tilde{\alpha}}{(1-\tilde{\alpha})(1-\kappa\gamma)}L_b(\tau) = \frac{\alpha}{(1-\alpha)}\frac{(1-\gamma)}{(1-\kappa\gamma)}L_b(\tau)$$

In particular, the fraction $\alpha(\tau)$ of good loans in the entire bank-τ portfolio converges to α, as κ converges to 1. More generally,

$$\alpha(\tau) \equiv \frac{1}{1+\frac{(1-\alpha)}{\alpha}\frac{(1-\kappa\gamma)}{(1-\gamma)}} = \frac{\alpha}{1+\frac{(1-\alpha)\gamma}{1-\gamma}(1-\kappa)} < \alpha$$

for these banks. Define the coefficient

$$\tilde{C}(\tau,p) = \frac{\tilde{\alpha}}{(1-\tilde{\alpha})(1-\kappa\gamma)}\gamma(d_g - rp(1-\tau)) + \gamma(d_b - rp + Rp\tau) \tag{41}$$

Then

$$L_b(\tau) = \frac{1}{1 - \frac{1-\tilde{\alpha}}{\tau p}\tilde{C}(\tau,p)} f(\tau) \tag{42}$$

and

$$X(\tau;p,p) = \tilde{C}(\tau,p)L_b(\tau) + f(\tau) \tag{43}$$

For $\tau > \tau^*$, all of $X(\tau;p,p)$ will be invested in primary loans. We obtain

$$\begin{aligned} L_g(\tau) &= \gamma\kappa L_g(\tau) + \frac{\alpha}{1-(1-\tau)p}X(\tau;p,p) && (44)\\ L_b(\tau) &= \frac{1-\alpha}{1-(1-\tau)p}X(\tau;p,p) && (45)\end{aligned}$$

implying

$$(1-\alpha)(1-\kappa\gamma)L_g(\tau) = \alpha L_b(\tau)$$

or

$$L_g(\tau) = \frac{\alpha}{(1-\alpha)}\frac{1}{(1-\kappa\gamma)}L_b(\tau)$$

Therefore

$$\alpha(\tau) \equiv \frac{1}{1 + \frac{(1-\alpha)(1-\kappa\gamma)}{\alpha}} = \frac{\alpha}{1-\kappa\gamma(1-\alpha)} > \alpha$$

for these banks. Define the coefficient

$$C(\tau,p) = \frac{\alpha}{(1-\alpha)}\frac{1}{(1-\kappa\gamma)}\gamma(d_g - rp(1-\tau)) + \gamma(d_b - rp + Rp\tau) \tag{46}$$

Then

$$L_b(\tau) = \frac{1}{1 - \frac{1-\alpha}{1-(1-\tau)p}C(\tau,p)} f(\tau) \tag{47}$$

and

$$X(\tau;p,p) = C(\tau,p)L_b(\tau) + f(\tau) \tag{48}$$

To solve for the equilibrium price, one needs to equate supply and demand for loans. Market supply is given by

$$S(p) = (1-\gamma\alpha)\gamma L(p)$$

Market demand is given by

$$D(p) = \frac{1}{p}\int_{\tau\in\Psi}\frac{1}{\tau}X(\tau;p,p)d\tau$$

Proposition 4. *Demand is a decreasing function of* p*. There is a unique* p*, equating supply and demand.*

Proof. TO BE COMPLETED. A sketch of the argument is as follows. First, $\tau^*(p)$ is a decreasing in p (Proposition 3) and hence Ψ becomes smaller as p decreases. Furthermore, $X(\tau; p, p)$ is also decreasing in p and the first result follows. The next step is to show that supply is increasing in p, $D(0) > S(0)$, and $D(1) < S(1)$ implying the second result. □

4.3 Date $t = 0$ and cash-in-the-market pricing.

At date $t = 0$, banks get caught in a possibly overleveraged situation: their asset position has been calculated, assuming that the price p remains at the "sunny equilibrium" price $p = 1$. In period $t = 0$, there will be banks that need to declare bankruptcy, while other banks will need to sell loans of both types in order to repay their short-term lenders.

To simplify the calculation of the equilibrium, we continue to assume maximal leverage:

Assumption 4. *In period* $t = 0$*, banks find it in the interest of the share holders to obtain maximal leverage, as they go forward.*

Let the period equilibrium price be p_0. We note

Lemma 3.

$$p_0 < 1$$

Proof. At $p_0 = 1$, equal resources are needed for investing in a new loan or investing on the secondary market. Furthermore, banks can continue with the leverage as in the "sunny" equilibrium. Some banks will therefore find it in the interest of their shareholders to sell some of the bad loans at $p_0 = 1$. But then, the fraction of the good loans on the secondary market is below α, the mix of good loans on the primary market. In that case, it is strictly better to only generate loans on the primary market, a contradiction. □

Indeed, the construction of the equilibrium below can be generalized also to the "sunny" equilibrium: there is a multiplicity of equilibria there in any period, given that all banks enter at maximal leverage for $p = 1$ from the previous period. What is special here is that the equilibrium with $p = 1$ can be ruled out per the lemma above, i.e. the cash-in-the-market equilibrium must prevail.

Due to full leverage for the pre-crisis period price of $p_{-1} = 1$, a bank of type τ can generate equity $e(\tau; p_0, 1)$ per unit of loan from selling it in the market, rather than releveraging it again, given by

$$e(\tau; p_0, 1) = d + \kappa p_0 - R(1 - \tau) \tag{49}$$

where $d = \alpha d_g + (1-\alpha)d_b$ was defined in (5). Note that all banks enter period $t = 0$ with the unconditional mix of good and bad loans. Note, though, that $e(\tau; p_0, 1)$ might be negative: in that case, the bank sells all assets and declares bankruptcy, repaying only a partial amount of the short-term loan. Further note, that a bank that sells a fraction ϕ of its non-terminating assets, while obtaining another short-term loan $(1-\phi)\kappa p_0(1-\tau)$ generates additional equity

$$\begin{aligned} e_\phi(\tau; p_0, 1) &= d + \phi\kappa p_0 - (R - (1-\phi)\kappa p_0)(1-\tau) && (50) \\ &= d + \phi\kappa p_0 \tau - (R - \kappa p_0)(1-\tau) && (51) \end{aligned}$$

A bank can feasibly only do this, if $e_\phi(\tau; p_0, 1) \geq 0$: if strictly larger than zero, the bank can then invest it on the primary or the secondary market. In particular, a bank that does not sell any assets generates additional equity (per loan) of

$$e_0(\tau; p_0, 1) = d - (R - \kappa p_0)(1-\tau)$$

This is only feasible for banks with low leverage, i.e. a high τ.

We shall concentrate the analysis on the following case.

Assumption 5. *Cash-in-the-market pricing. The equilibrium price p_0 is so low, that banks prefer to hold on to their bad-quality loans rather than selling them.*

I.e., a bank will not sell loans, unless it is forced to due to its low equity position. As a consequence, the primary will cease to function, as it is strictly better to purchase assets on the secondary market.

One can now distinguish the following cases.

1. Bankruptcy. This case occurs for

$$\tau \leq \tau_A = 1 - \frac{d + \kappa p_0}{R}$$

In that case, the bank sells all assets. Therefore, the assets sold by all banks of type τ, including the banks chosen for exit, is

$$S(\tau) = \kappa L(\tau)$$

where $L(\tau)$ is given by equation (7). Moreover, the amount of good and bad loans offered by these banks are

$$\begin{aligned} S_g(\tau) &= \alpha\kappa L(\tau) \\ S_b(\tau) &= (1-\alpha)\kappa L(\tau) \end{aligned}$$

Note that $\tau_A < 0$ per assumption 1, if $p_0 = 1$ as in the "sunny" equilibrium.

2. Emergency sales of bad and good loans. In this case, only selling the bad assets, $\phi = 1 - \alpha$, would result in non-positive equity. Examining (51), this occurs for

$$\tau_A < \tau \leq \tau_B = 1 - \frac{d + (1-\alpha)\kappa p_0}{R - \alpha\kappa p_0}$$

In that case, the banks will sell all bad assets. But they need to sell a fraction $\phi > 1-\alpha$

The total supply on the market is therefore given by

$$S = S_g + S_b \tag{53}$$

(where buyers cannot see whether a loan is good or bad), with

$$S_g = \int_0^1 S_g(\tau) d\tau \tag{54}$$
$$S_b = \int_0^1 S_b(\tau) d\tau \tag{55}$$

Note that the supply depends on the market price p_0. The supply is actually downward sloping in the price.

Lemma 5. *Let $\underline{p}$ be the highest price consistent with banks holding on to their bad loans rather than selling them, if they can. The supply of loans is decreasing in the price on the range $p \in [0, \underline{p}]$.*

Proof. Sales of exiting banks are exogenous. The endogenous sales are driven by the need to repay the short-term lenders, i.e. by the need to raise a certain amount of cash. The lower the price, the more assets are required to raise a required amount. □

One can also calculate the fraction of good assets in the market,

$$\tilde{\alpha}_0 = \frac{S_g}{S} \tag{56}$$

We have

Lemma 6.

$$\tilde{\alpha}_0 \leq \alpha$$

The inequality is strict, if there is a nonzero mass of banks with $\tau_A < \tau \leq \tau_C$.

Proof. Immediate. □

The demand for assets is given by the leveraged demand of all continuing banks with $\tau > \tau_C$ as well as the new equity $f(\tau)$. These banks will purchase all of their equity in the secondary market: we assume that they do so at maximal leverage. I.e., we have

$$D = \int_0^1 \frac{1}{p_0 \tau} f(\tau) d\tau + \gamma \int_{\tau_C}^1 \frac{1}{p_0 \tau} (d - (R - \kappa p_0)(1 - \tau)) L(\tau) d\tau \tag{57}$$

Note that τ_C is decreasing in the price, i.e., the cash demand on the secondary market by the surviving banks increases with the market price. Whether demand can be upward sloping or must be downward sloping therefore requires additional analysis or particular numerical values.

The price p_0 can now be determined per equating supply and demand. Since demand and supply may not have "traditional" slopes, it is conceivable that there are multiple equilibria. We shall leave that to the numerical analysis.

Finally, one obtains the "initial" condition for the subsequent adverse-selection dynamics, given in subsection 4.1. The loans $L_g(\tau)$ and $L_b(\tau)$ carried forward to the next period need to be calculated according to the cases above.

1. Bankruptcy range, i.e.
$$\tau \leq \tau_A$$
In that case, only fresh equity results in loan holdings,
$$\begin{aligned} L_{g,1}(\tau) &= \frac{\tilde{\alpha}_0}{p_0 \tau} f(\tau) \\ L_{b,1}(\tau) &= \frac{1-\tilde{\alpha}_0}{p_0 \tau} f(\tau) \end{aligned}$$

2. Emergency sales of bad and good loans, i.e.
$$\tau_A < \tau \leq \tau_B$$
Thus,
$$\begin{aligned} L_{g,1}(\tau) &= \frac{\tilde{\alpha}_0}{p_0 \tau} f(\tau) + \gamma(1-\phi(\tau))\kappa L(\tau) \\ L_{b,1}(\tau) &= \frac{1-\tilde{\alpha}_0}{p_0 \tau} f(\tau) \end{aligned}$$

3. Emergency sales of bad loans only, i.e.
$$\tau_B < \tau \leq \tau_C$$
Then,
$$\begin{aligned} L_{g,1}(\tau) &= \frac{\tilde{\alpha}_0}{p_0 \tau} f(\tau) + \gamma\alpha\kappa L(\tau) \\ L_{b,1}(\tau) &= \frac{1-\tilde{\alpha}_0}{p_0 \tau} f(\tau) + \gamma(1-\alpha-\phi(\tau))\kappa L(\tau) \end{aligned}$$

4. Investing in the secondary market, i.e.
$$\tau > \tau_C$$

Here,

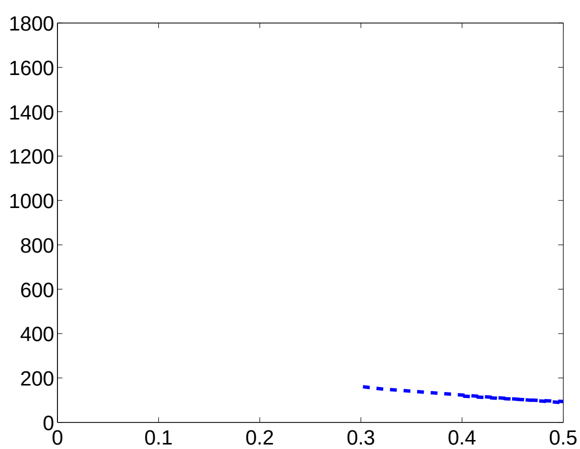
$$L_{g,1}(\tau) = \frac{\tilde{\alpha}_0}{p_0\tau}\Big(f(\tau) + (d - (R - \kappa p_0)(1-\tau))\gamma L(\tau)\Big) + \gamma\alpha\kappa L(\tau)$$
$$L_{b,1}(\tau) = \frac{1-\tilde{\alpha}_0}{p_0\tau}\Big(f(\tau) + (d - (R - \kappa p_0)(1-\tau))\gamma L(\tau)\Big) + \gamma(1-\alpha)\kappa L(\tau)$$

With these initial conditions, one can now proceed to the analysis in 4.1.

5 A numerical example.

For the numerical examples, we assume that the creation of new banks is given by a uniform density on the interval $\tau \in [0.05, 0.5]$, integrating to unity, $F = 1$. The other parameters and some implications can be seen from table 1.

Figures 2 and 3 show the equilibrium in the "sunny" regime. There is no adverse selection and the equilibrium price is $p = 1$. The loan portfolio and the stock market price per unit of equity is the higher, the higher is the allowable leverage, i.e., the lower is τ.



α	0.9
γ	0.95
R	1
κ	0.9
$\underline{\tau}$	0.05
ϵ	0.0013
d_g	0.113
d_b	0
d_s	$= \bar{d} = \alpha d_g + (1-\alpha) d_b$
$d_s = \bar{d}$	0.101
r	0.100
$\tilde{\alpha}$	0.310
$\tilde{\alpha}_{\text{prim}}$	0.984
$\tilde{\alpha}_{\text{sec}}$	0.756
p_0	0.044
$p_{(rainy)}$	0.839
τ^*_{rainy}	0.384
L_{sunny}	131.806
$L_{\text{rainy}}/L_{\text{sunny}}$	77 %
$E[q_{\text{sunny}}]$	1.173
$E[q_{\text{rainy}}]$	1.206
$E[q_{\text{rainy}}]/E[q_{\text{sunny}}]$	1.028

Table 1: Parameters for the numerical example and some results.

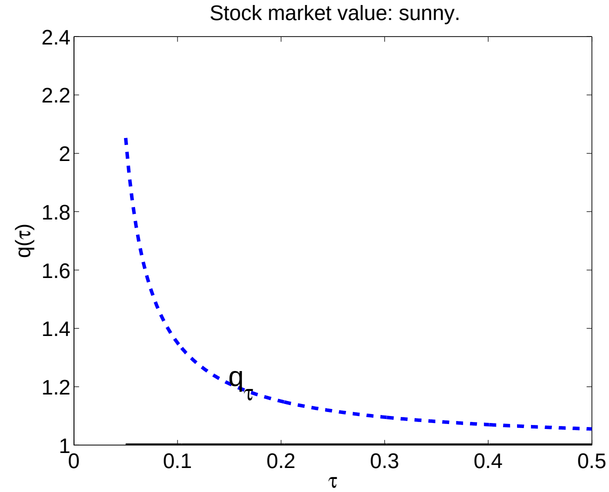


Figure 3: *The value of a unit of equity to outside investors in the "sunny" regime. This "stock market value" is the higher, the higher is the allowable leverage, i.e. the lower is* τ.

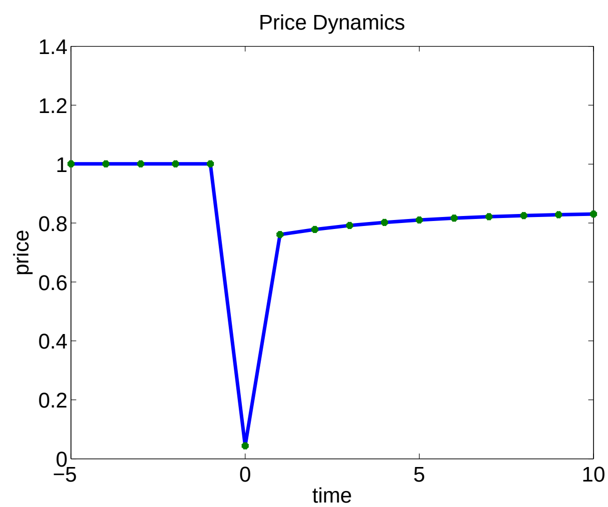


Figure 4: *The conjectured price dynamics of the model. The steady state values for the sunny regime* ($t < 0$) *and the rainy regime* ($t > 0$) *as well as the price and loan volumes at date* $t = 0$ *have been calculated numerically, while the dynamics for* $t > 0$ *has been "filled in", assuming exponential convergence from a level slightly below the final steady state.*

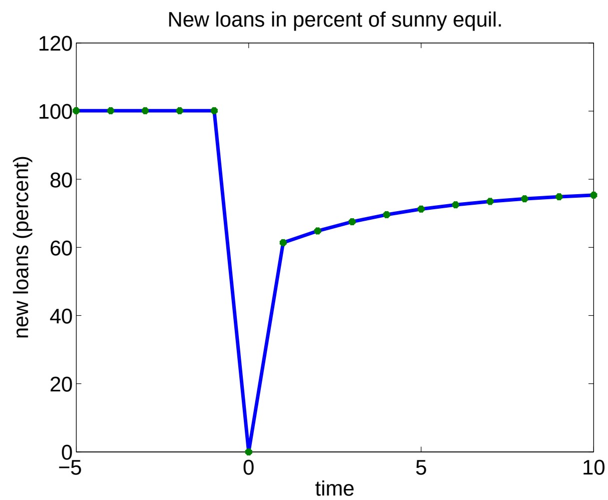


Figure 5: *The conjectured new loan dynamics of the model. The steady state values for the sunny regime ($t < 0$) and the rainy regime ($t > 0$) as well as the price and loan volumes at date $t = 0$ have been calculated numerically, while the dynamics for $t > 0$ has been "filled in", assuming exponential convergence from a level slightly below the final steady state.*

the final steady state has been assumed. A full numerical calculation is planned for a future version of this paper.

Figures 6 and 7 show the situation at date $t = 0$: while the first figure shows the calculation of the equilibrium price per intersecting demand and supply, the second figure shows the threshold for the various types of banks. The cash-in-the-market pressure forces banks to liquidate their portfolio, and this assumption has been maintained in this figure at all prices. While there are two intersections of supply and demand in figure 6, this assumptions fail at the higher price, i.e., the higher price cannot be an equilibrium and the lower price is the unique equilibrium here.

To understand the forces at work in the "rainy" adverse selection situation, figure 8 examines the mechanics of leverage in the model, comparing (1) to (3) at a hypothetical equilibrium price of $p = 0.5$ (which is actually considerably below the equilibrium price of the numerical example). I.e., the figure compares, how many loans can be held, if a type-τ bank is active on the primary market or the secondary market, and arbitrarily assuming an equilibrium price $p = 0.5$. In the "sunny" regime, the market price is $p = 1$, and there would not be a difference between these two lines: the differences arise only in the "rainy" adverse selection scenario.

Figure 9 and 10 show the determination of the steady state equilibrium price in the "rainy" regime. While the first figure shows the intersection of the (now upward sloping) supply and demand, the second figure shows how banks switch from the secondary (blue) to

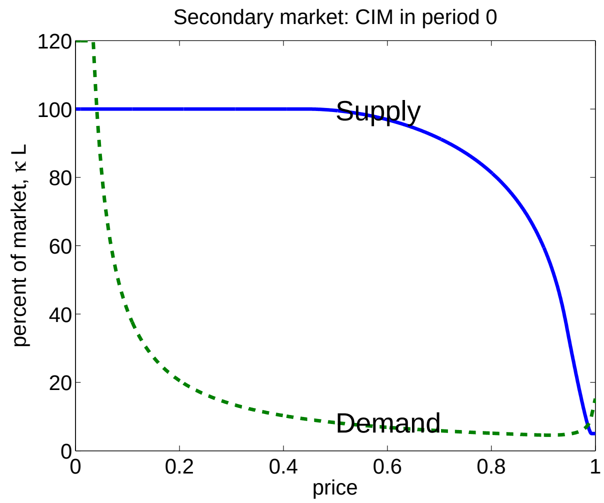


Figure 6: *The calculation of the equilibrium price per intersecting demand and supply at date $t = 0$. Note how supply is actually downward sloping, as a function of the price. The cash-in-the-market pressure forces banks to liquidate their portfolio, and this assumption has been maintained in this figure at all prices. This assumption fails at the higher prices, and therefore there is actually a unique equilibrium at the left intersection of demand and supply and the lower price.*

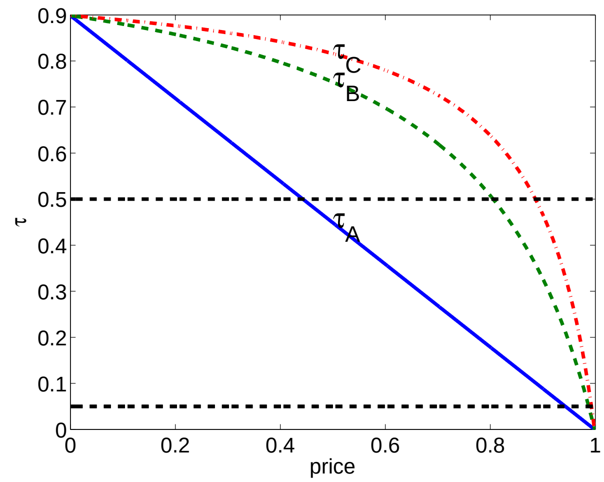


Figure 7: *Threshold for the various types of banks at date $t = 0$, as a function of the secondary market price for loans.*

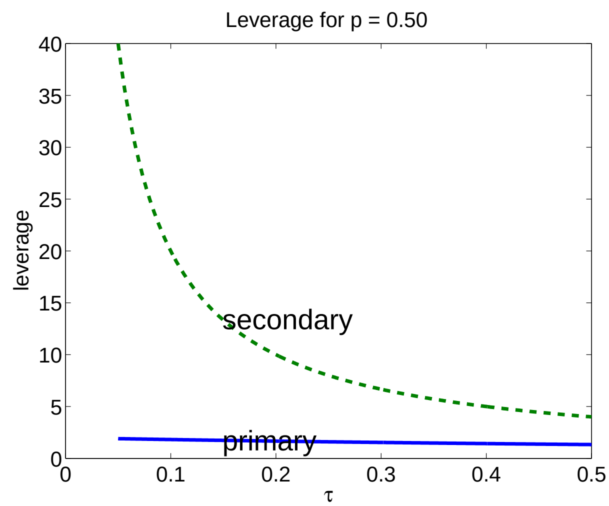


Figure 8: *The figure examines the mechanics of leverage in the model and compares, how many loans can be held, if a type-τ bank is active on the primary market or the secondary market, and arbitrarily assuming an equilibrium price* $p = 0.5$.

the primary (red) market as the secondary market price rises, given a particular type τ.

Figures 11 and 12 show the "rainy" steady state distribution of loan portfolios and stock market values, as a function of leverage type τ, and correspond to their "sunny" regime counterparts, i.e. figures 2 and 3. Note that the stock price is still rising in allowable leverage, i.e. falling in τ. However, the loan portfolio jumps up at a certain point, as τ is increased: at this point, banks engage in the primary rather than the secondary market. I.e., the numerical example has the feature that it is now the low-leverage rather than the high-leverage banks, which are large in terms of the market share. This result may be rather counterintuitive: loan portfolios are built up from retained profits, so it looks as if high-τ banks are more profitable, per engaging only on the primary market. But then, as low-τ banks are allowed to do what high-τ banks do, so why do they not simply "imitate" low-leverage (high-τ) banks? The resolution of this puzzle lies in the distinction between the steady state distribution and the discounting of future profits, i.e., what is relevant for a bank is not how large the loan portfolio will eventually be, but how long it takes to get there. Appendix A provides a simplified example to illuminate this phenomenon.

6 Final Remarks

How did a modest increase in subprime US mortgage default rates turn into a world-wide financial crisis? Why did prices for collateralized debt assets decline so sharply, and, relatedly,

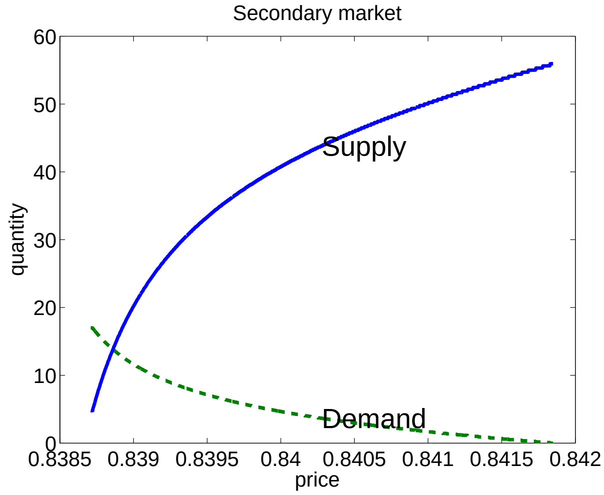
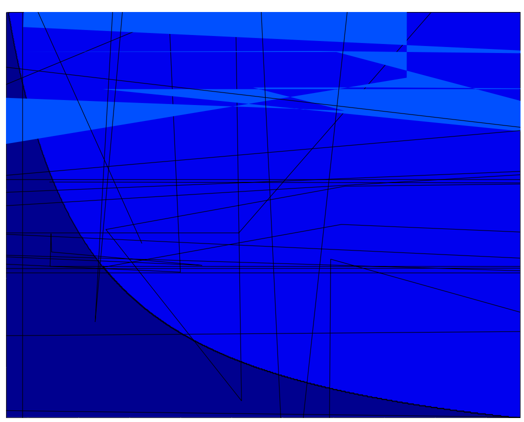


Figure 9: *Determination of the steady state equilibrium price in the "rainy" regime, per the intersection of supply and demand, as a function of price.*



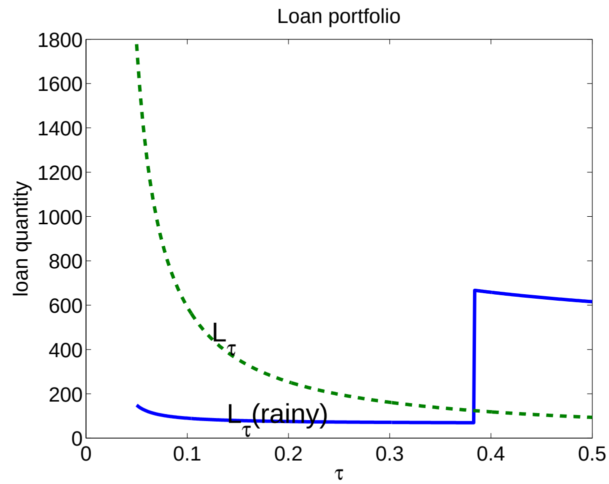


Figure 11: *The "rainy" steady state distribution of loan portfolios and stock market values, as a function of leverage type τ. Note that the loan portfolio jumps up at a certain point, as τ is increased: at this point, banks engage in the primary rather than the secondary market. Appendix A provides a simplified example to illuminate this phenomenon.*

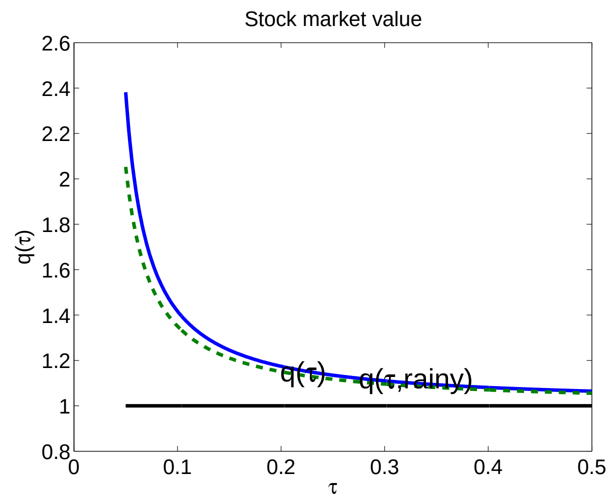


Figure 12: *The value of a unit of equity to outside investors in the "rainy" regime. This "stock market value" is the higher, the higher is the allowable leverage, i.e. the lower is τ.*

why did the lending market and resale market for these securities dry up? How did the breakdown in these markets lead to the near collapse of the banking sector and to a significant decline in real loan activity?

This paper answers these questions in a simple model of short-term repo-financing and adverse selection, in which a crisis in the loan market follows an initial "cash-in-the-market" crash, after a part of the loan market is revealed to be worse than the rest. Banks issue long-term loans and finance them with repurchase agreements ("repos") from short-term lenders in order to leverage up their equity. Banks differ in their leveraging ability. Aside from the primary loan market ("loan-generation"), there is a secondary market, where loans can be traded among banks and sold by short-term lenders. Loans differ in quality, but the quality is observable only to the bank currently holding the loan contract. Therefore, the secondary loan market suffers from adverse selection, leading to important repercussions in loan origination. At some random time, the loan pool unexpectedly becomes heterogeneous. This leads to an initial crisis where highly levered banks are forced to sell loans, flooding the market and leading to fire-sale prices.

I have examined the resulting equilibrium with a numerical example. In that example, the initial date-$t = 0$ "cash in the market" crisis is followed by a prolonged period of subdued secondary loan prices, bank equity prices, and lending activity, in which the low-leverage banks rather than the high-leverage banks dominate the market.

References

Acharya, Viral, Phillip Schnabl, and Gustavo Suarez, "Securitization without Risk Transfer," *NBER Working Paper No. w15730*, 2010.

Brunnermeier, Markus, "Deciphering the Liquidity and Credit Crunch 2007-2008," *Journal of Economic Perspectives*, 2009, *23(1)*, 77–100.

Cheng, Inh-Haw, Harrison Hong, and Jose Scheinkman, "Yesterday's Heroes: Compensation and Creative Risk Taking," *Working Paper*, 2009.

Gorton, Gary, "Slapped in the Face by the Invisible Hand: Banking and the Panic of 2007," *Working Paper*, 2009.

Krishnamurthy, Arvind, "How Debt Markets Have Malfunctioned," *Journal of Economic Perspectives*, 2010, *24*, 3–28.

Appendix

A Understanding the shift of loans towards low-leverage banks.

In the numerical example, it may seem puzzling that the distribution of loans can shift towards the high-τ banks. However, this is not so as the following simplified version of the model demonstrates.

Consider a bank, discounting the future at return $R > 1$. At the beginning of every period, and iid across time with probability $(1 - \gamma)$, the bank has to cash in all its assets and deliver this as payment to the shareholders: indeed, this is the only way shareholders get paid a dividend. Also at the beginning of every period (but slightly after a possible cash-out), it receives one unit of fresh cash (or equity) every period from its shareholders. The bank has the choice between two investment projects:

1. Project "Primary". In that case, one unit of cash is turned into ν units of the project. From one period to the next, the project grows with the factor ϕ, i.e. after one period, the project has size ν, after two periods, it has size $\phi\nu$, in the third it has size $\phi^2\nu$ and so forth, until the bank needs to cash everything in.

2. Project "Secondary". In that case, one unit of cash is turned into λ units of the project. From one period to the next, the project grows with the factor σ, i.e. after one period, the project has size ν, after two periods, it has size $\sigma\lambda$, in the third it has size $\sigma^2\lambda$ and so forth, until the bank needs to cash everything in.

Assumption 6. *I assume that $\lambda > \nu > 0$ and $1 < \sigma < \phi < 1/\gamma$.*

I.e, the primary project is initially smaller, but then grows faster over time, and both projects grow more slowly than the cash-in risk. I furthermore assume that the price for one unit of each project is always p, i.e., when the bank cashes in all its assets, it will pay its shareholders the number of units of projects it currently owns (and the proceeds re-investing the newly injected cash) times this price p. I finally assume that the assets are dissolved when they are cashed in[8]

[8]That may superficially seem like a potentially important difference to the original paper. It is needed here to get a stationary distribution, see the end of the note, though this could be finessed away by also introducing a probability of death κ for the projects. More importantly and with this additional death probability, the demand for project units always outstrips the supply of old project units, so that there is always asset creation. It then is rather immaterial, whether I assume a secondary market for old assets, to be "filled up" with the creation of new assets, or whether all assets are assumed new, as I do here.

It is easy to calculate the net present value of investing in either project. If the bank invests in the primary project, then

1. with probability $(1-\gamma)$ and discounted with R, it will have to cash in within one period, delivering ν units of the project or $p\nu$ units of cash to its shareholders.
2. with probability $\gamma(1-\gamma)$ and discounted with R^2, it will have to cash in at the beginning of the second period, delivering $\nu\phi$ units or $p\nu\phi$ units of cash.
3. etc.

with the sum total

$$\begin{aligned} v_{\text{prim}} &= \frac{(1-\gamma)\nu p}{R} \sum_{j=0}^{\infty} \left(\frac{\gamma\phi}{R}\right)^j \\ &= \frac{(1-\gamma)\nu p}{R} \frac{1}{1-\frac{\gamma\phi}{R}} \\ &= \frac{(1-\gamma)\nu p}{R-\gamma\phi} \end{aligned} \tag{58}$$

Note that this is the result of one unit of cash infusion. To calculate the value of the bank (and given that there is equity infusion every period), one needs to multiply that with $R/(R-1)$: this does not change anything about the subsequent analysis.

Likewise, the net present value of the secondary project from a one-time one-unit cash infusion is given by

$$v_{\text{sec}} = \frac{(1-\gamma)\lambda p}{R-\gamma\sigma} \tag{59}$$

As a result, I have

Proposition 5. *The secondary project has a higher net present value, $v_{sec} > v_{prim}$, iff*

$$\frac{\lambda}{\nu} > \frac{R-\gamma\sigma}{R-\gamma\phi} \tag{60}$$

The interpretation of (60) is straightforward: the relative initial size ratio must outweigh the relative importance of future growth, when discounted at factor R, in order for the secondary project to be more attractive.

Suppose now, however, that there actually is no choice between these two projects, but rather, that "primary banks" (read: high-τ-banks) always invest in the primary project and "secondary banks" (read: low-τ-banks) always invest in the secondary project. Suppose there are many such banks, and that the aggregate new cash infusion every period for each type of bank is unity. Assume that the cash-in risk is iid across banks. Appealing to a law of large numbers, there is therefore a fraction $1-\gamma$ of the projects that are cashed in (and dissolved, see the footnote) every period. With this, I can calculate the total volume of outstanding project units in each of the two banking sectors in steady state. For the "primary" sector, that total volume must satisfy that it equals the growth of the old remaining projects plus the creation of new projects from new cash infusion, i.e. must satisfy the equation

$$L_{\text{prim}} = \gamma\phi L_{\text{prim}} + \nu$$

or

$$L_{\text{prim}} = \frac{\nu}{1-\gamma\phi} \tag{61}$$

Likewise, for the "secondary" sector, I get

$$L_{\text{sec}} = \frac{\lambda}{1-\gamma\sigma} \tag{62}$$

A simple calculation shows

Proposition 6. *The secondary sector has a higher volume, $L_{sec} > L_{prim}$, iff*

$$\frac{\lambda}{\nu} > \frac{1-\gamma\sigma}{1-\gamma\phi} \tag{63}$$

The equation (63) looks similar to (60) and has likewise a straightforward interpretation. The key difference arises from the discounting inherent in the forward-looking calculations of (60) relative to the steady-state-maintenance (and therefore no discounting) in equation (63). To gauge the effect of this difference, I note

Proposition 7. *Define the function $f(x)$ per*

$$f(x) = \frac{x-\gamma\sigma}{x-\gamma\phi} \tag{64}$$

The function is decreasing in $x > \gamma\phi$. Furthermore, $f(1) > 1$.

Putting this all together, I have the following

Theorem 1. *There exist $\lambda > \nu > 0$, such that (60) is satisfied, but that (63) is violated. In words, there exist parameter constellations, so that secondary projects are more valuable, but that the volume of primary projects is larger.*

Proof. Find $\lambda > 0$ and $\nu > 0$ such that

$$f(1) > \frac{\lambda}{\nu} > f(R)$$

Since $f(x)$ is decreasing, this is possible. Furthermore, since $f(1) > 1$, it is possible to find λ and ν such that $\lambda > \nu$. $\square$

The possibility provided by the theorem above is precisely the "puzzling" situation emerging in the numerical example of the paper.