Sorting and Factor Intensity: Production and Unemployment across Skills*

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Abstract

When firms choose the allocation of workers, they can adjust not only the type of worker, the extensive margin, but also the intensive margin, how many of those worker to employ. We propose a tractable matching model with such factor intensity. Positive sorting arises if a simple cross-margin-complementarity condition holds: within-complementarities in extensive and intensive margin have to exceed the between-complementarities across intensive and extensive margin. We characterize the equilibrium allocation, wages and factor intensities. In an immediate extension that allows for frictional hiring we also analyze the presence of unemployment across types. Unemployment is decreasing in skills, and we find conditions under which firm size is increasing in skill.

Keywords. Factor intensity. Unemployment. Competitive Search Equilibrium. Two-Sided Matching. Complementarity.

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1 Introduction

We propose a theory of labor market matching where firms can adjust the factor intensity. When aggregate conditions change, for example due to business cycles, international competition or changes in factor endowments, firm react not only by adjusting the extensive margin (the skill of a worker that is right for the job), but also the intensive margin (how many of those workers to employ). The intensive margin indicates how labor productivity changes. For example, during a recession there can be both increased unemployment *and* growing labor productivity. Firms adjust capacity by laying off workers, which in the presence of a concave production technology raises the marginal product of the workers.

Key in understanding the interplay of productivity and the intensive margin is heterogeneity in skills. When workers differ in skills and ability, and firms employ technologies with different levels of productivity. In standard models of sorting the relevant choice is on the extensive margin by choosing the optimal allocation over different types of skills. Here, firms or technologies simultaneously operate two margins: not only do they optimally choose the skill of a worker, they also choose how many of those skilled workers to employ. The ensuing capital-labor ratio will as a result vary across skills and technologies. The combination of two-sided heterogeneity and endogenous factor intensity pins down the equilibrium outcome and determines wage formation, wage inequality and unemployment.

Unlike the standard Beckerian matching framework, firms are large and we can analyze changes in the firm size and composition. In the presence of an aggregate productivity shock for example, firms are able to adjust the skill quality as well as the number of those skilled workers. Even though our model is static, in a dynamic setting, the latter may turn out to be more easily adjusted since the adjustment is for marginal types only, rather than the entire pool.

Our setting also permits us to analyze equilibrium unemployment. In a directed search environment, firms choose their skill composition and size taking into account frictions in the hiring process. This permits us to analyze how unemployment changes across skills and firm size. Interestingly, in our framework long term unemployment arises naturally and it is determined characteristics of the production technology. In particular, firms might compete for higher skilled workers rather than substituting them with low skilled ones, thus leaving the low skilled permanently unemployed. Interestingly, aggregate productivity shocks will lead to the low skilled to move in and out the state of unemployability, while simultaneously equilibrium unemployment adjusts for higher-skilled workers.

Our main findings. First, we show a surprisingly simple condition for positive sorting. As in the one-to-one matching model, it requires a strong enough degree of complementarity between the type of firm and that of the worker. Now the strength also depends on the marginal change of output in labor and technology intensity and how that varies across worker and firm types respectively. The relevant condition for sorting therefore has to involve changes in the quality dimensions (extensive margin) and in the quantity of factor inputs (intensive margin). Positive assortative matching arises if the the total degree of intensive and extensive complementarity exceed the cross-complementarities between the intensive and extensive margin. It might be interesting to note the latter become unimportant when the intensive margin is Lientieff, in which case standard supermodularity prevails. Second, even under assortative matching one still has to determine the exact allocation and the associated factor intensities. We derive these expressions, including conditions for them to be increasing. Finally, we integrate labor market frictions into the model and show how unemployment varies accross worker types. Also, we show how the condition is extended when additional generic capital inputs can be bought in the world market, and when the goods have to be sold in an output market characterized by monopolistic competition.

Related literature. The idea of modeling firms as many-to-one matchings has extensively been analyzed by Kelso and Crawford (1982). They propose a general, discrete agent framework. While it is well-known that the stable equilibrium of many-to-one matchings may not exist, Kelso and Crawford derive a sufficient condition for existence, that of gross substitutes. This condition basically means that adding one worker will not dramatically increase the productivity of all the other ones. This condition is satisfied in our setting since the intensive margin is concave: an additional worker is strictly less productive. While Kelso and Crawford allow arbitrary type-dependent production processes only restricted by the cross-substitute condition and show existence, characterization results under intensive margins are essentially missing. Our work is limited to the case where production depends only the type of capital and the capital intensity that is assigned to each worker. For our continuum economy this allows us to capture the intensive margin element while retaining a level of tractability that proves extremely useful in application.

The main difference of our approach to the standrad matching models that feature in many recent applications originating from Kantorovich (1942), Koopmans and Beckmann (1957), Shapley and Shubik (1971), Becker (1973), is that these earlier models restrict attention to settings where each agent on one side can only be matched with one agent on the other side, and there are interaction in the amount that can be produced by each pair. Models following Sattinger (1975) do not have any complementarities in the output that a pair produces, but different workers need different amounts of supervision which limits the number of workers that a particular firm can hire at some maximum value. In both cases the condition for sorting remains simple (supermodularity in the former, log-supermodularity in the latter) but the ratio at which each firm can hire workers is very restrictive. In particular, the firm cannot substitute additional workers.

Using a continuum economy, we allow any ratio of agents from one side relative to agents on the other side. Restricting the output to Liontief retrieves the condition of these earlier models, but in general it reveals a richer condition that takes into account the extensive margin in the assignment.

While some settings such as the marriage market one-on-one matching clearly seems to be the relevant case, examples of intensive margin matching abound. Our leading example is the assignment of workers to firms, where the number of workers per firm does not need to be unity, but can be freely chosen. Another example is an extension to the classical transportation problem going back to Kantorovich (1942). He considered goods that a produced at various plants and are stored at various locations, and unit costs of transportation that depend on the production site and the storage location. In his setting, each good needs to be stored in exactly one unit of storage, while our setup allows for the case where more goods can be cramped into each unit of storage but damage increases with storage intensity. We discuss this example as well as other applications such as matching between teams of workers at the end, but will focus on our leading example in the main body.

Our model differs from settings such as the Roy (1951) model and its recent variants in e.g. Heckman and Honore (1990) where each firm (or sector) can absorb unbounded numbers of agents. In our setup marginal product decreases as any particular firm gets extensively used. Some models combine the Roy model with a demand by consumers that entials a constant elasticity of substitution (CES), which implies that the price falls when more workers choose to work in a particular sector (see recently Costinot (2010)). The difference is that in such settings no agent internalizes the fact that the price falls when more output is produced. In our settings the firms understand that output falls when they produce more. In the final examples we also allow for a CES demand structure, but now this results in a model of imperfect competition similar to Dixit and Stiglitz (1977), only that now two-sided heterogeneity and an extensive margin are allowed.

Finally, the extension to search frictions is linked to recent developments on sorting in search markets by Shimer and Smith (2000), Shi (2001), Shimer (2005), Atakan (2006), and Eeckhout and Kircher (2010), but the real novelty of our approach to is to provide general conditions for large firms that can aquire labor either competitively or through a competitive search channel.

2 The Model

Players. The economy consists of heterogeneous firms and workers. Workers are indexed their skill $x \in X = [\underline{x}; \overline{x}]$; and $H_w(x)$ denotes the measure of workers with skills below x; with continuous non-zero density h_w. Also firms are heterogeneous in terms of some propriatory input into production that is exclusive to the firm, such as scarce managerial talent or particular propriatory capital goods. Firms are indexed by their productivity type $y \in Y = [\underline{y}; \overline{y}]$; where $H_f(y)$ denote the measure of firms with type below y; with non-zero continuous density h_f: It will be useful to think of the number of firms as small relative to the number of workers, which will allow each firm to hire a continuum of workers.[1]

Preferences and Production: Firms and workers are risk-neutral expected utility maximizers. If a firm of type y hires an amount of labor l_x of type x; it has to choose a fraction of its propriatory resources r_x that it dedicates to this worker type. This allows the firm to produce output

$$F(x; y; l_x; r_x)$$

with this worker type. In the production function the first two arguments $(x; y)$ are quality variables regarding firm productivity and worker skill, while the latter two arguments $(l_x; r_x)$ are quantity variables describing the level of inputs. The output is assumed to be increasing and twice differentiable in all arguments, and strictly concave in each of the quantity variables: For most analysis we also assume that output is constant returns to scale in the quantity variables, which turns it into a theory where the factor intensity r_x/l_x with which each worker is utilized becomes important. Other non-differentiated inputs into production can easily be accomodated, as we discuss at the end of this section.

The total output of a firm is the sum of the outputs accross all its worker types. A firm that produces with only one worker type produces

$$f(x; y; l) := F(x; y; l; 1):$$

This introduces an intensive margin into models that have traditionally focussed on pair-wise matching (see most literature following Becker (1973)), which is similar to imposing a one-unit labor force for all worker-firm pairs in this model. Since $f(x; y; l)$ is strictly decreasing in l; this theory provides a bridge

[1] Although the set of individuals has the same cardinality as the set of firms, it is helpful to think of the set of firms as a closed interval in $[\_ \ \bar{}\,] \subseteq \mathbb{R}$, and the set of workers as a two–dimensional subset $[\_ \ \bar{}\,] \times [0 \ 1] \subseteq \mathbb{R}^2$. When both sets are endowed with the Lebesgue measure, an active firm employs a continuum of workers, albeit of mass zero.

to the literature on large firms with decreasing returns (see e.g. the literature following Stole and Zwiebel (1996)) that usually analyzes homogeneous workers and firms. The tractability arises because quality complementarities are concentrated on the worker-firm interaction rather than on intra-worker skill-complementarities.

Competitive Market Equilibrium : In equilibrium, workers of type x obtain some expected utility $w(x)$ that coincides with the expected wage that they are paid. Firms take this hedonic schedule as given when they make their hiring decision. We will first fix ideas by outlining the definitions of an intensive-margin hedonic pricing equilibrium without search frictions, and then handle the important case with search frictions and associated unemployment in an extension along the lines of the competitve search literature.

Firm optimality in a frictionless competitive market means that a firm of type y maximizes its output minus wage costs as follows:

$$\max_{l_x, r_x} \int [F(x;y;l_x;r_x) - w(x)l_x]dx \tag{1}$$

where r_x can be any probability density function over x. Factoring out r_x from the square bracket reveals that the interior depends only on the factor intensity $= l_x/r_x$; which can be freely chosen at any level in $\Theta = \mathbb{R}_+$ by adjusting the labor input appropriately. Optimality requires that the firm places positive resources only on combinations of $x \in X$ and $\in \Theta$ that solve[2]

$$\max_{x,\theta} f(x;y; \ ) - \ w(x): \tag{2}$$

If there is only one such combination that solves this maximization problem, then the firm will hire only one worker type, allocate all resources to this type, and hire an amount of labor $l = \ :$

Feasibility of the allocation implies that firms attempt to hire no more workers than there are in the population. Denote by $\mathcal{R}(x;y; \ )$ the resource allocation in the economy, which describes the amount of resources that firms with a type below y devote to workers of a type below x that are employed with a factor intensity $l_x/r_x \leq \ :$ Let $\mathcal{R}(y|X;\Theta)$ denote the marginal over y when the other two variables can take any value in their type space: It denotes the amount of resources used by firms with type below y: Since the resources of each firm are normalized to one, this has to equal the amount of firms in the population, so feasibility requires $\mathcal{R}(y|X;\Theta) = H^f(y)$ for all y: Moreover, let $\mathcal{R}^\theta(\ |x;Y)$ denote the marginal with respect to of the distribution conditional on a particular worker type x: It denotes the

[2] Problem (1) is equivalent to $\max_{r_{(.)}} \int (\ _x \max_{\theta_x} [\ (\ \ d_x \ 1) - \ (\ )d_x])$, where $d_x = \ _x \ _x$ can be adjusted through appropriate hiring of workers. Clearly, resources are only devoted to combinations of and d that maximize (2).

resources spent by all firms on workers of with type x employed with intensity less than : Feasibility requires $\int \ \mathsf{d}\mathcal{R}(\ |\mathsf{x};\mathsf{Y}) \leq \mathsf{h}_w(\mathsf{x})$: It states that the amount of workers of type x demanded across all firm types cannot exceed the number of such workers in the population (where the labor demand is the factor intensity times the amount of resources allocated at this factor intensity).

Definition 1 *An intensive-margin hedonic pricing equilibrium is a tuple* (w,$\mathcal{R}$) *consisting of a non-negative hedonic wage schedule* $\mathsf{w}(\cdot)$ *and a resource allocation* $\mathcal{R}$ *such that*

1. Optimality: $(\mathsf{x};\mathsf{y}; \ ) \in \text{supp}\mathcal{R}$ only if it satisfies (2).

2. Market Clearing: $\int \ \mathsf{d}\mathcal{R}(\ |\mathsf{x};\mathsf{Y}) \leq \mathsf{h}_w(\mathsf{x})$, with equality if $\mathsf{w}(\mathsf{x}) > 0$:

Assortative Matching: Let $\mathcal{R}(\mathsf{x};\mathsf{y}|\Theta)$ be the marginal distribution of $\mathcal{R}$ over the firm and worker types at any level of intensity. It denotes the amount of resources devoted by firms with type below y to workers of skill below x: Matching is assortative if there exists a monotone function (x) and such that the support of $\mathcal{R}(\mathsf{x};\mathsf{y}|\Theta)$ only includes points $(\mathsf{x}; \ (\mathsf{x}))$: We call the assignment function. Matching is (strictly) positive assortative if the assignment function is (strictly) increasing, and it is (strictly) negative assortative if the assignment function is (strictly) decreasing. In the following we will restrict attention to assortative matching that can be supported by a differentiable assignment function:

Clarification of Market Structure and Integration of Different Resource Constraints, Generic Capital, and Competitive Search Frictions: In our exposition we assume that firms own a unit measure of a scarce resource and allocate it to the different workers that they hire. As in other matching models, it turns out that exactly the same allocations arise if each worker would buy the resources he wants to use in production. In this case the workers production function would be $\mathsf{F}(\mathsf{x};\mathsf{y};1;\mathsf{r})$; and his return decreases as he buys additional resources. In this case the equilibrium price for a given resource equals the profit that firm obtains in our exposition. Similarly, if firms are not endowed with any resources but have to buy both resources and labor, the same allocation arises but firms make zero profits since all profits accrue to the owners of the scarce resources.

Even within our exposition the production function F can be interpreted in boader terms. First, we interpreted r as the fraction of the firm's resources, implicitly using a unit measure of resources for each firm. This is just a normalization. If firms of type y have $\mathsf{T}(\mathsf{y})$ resources and produce $\tilde{\mathsf{F}}(\mathsf{x};\mathsf{y};\mathsf{l};\mathsf{t})$ by using t units of them, we can express this in terms of the fraction r of their resources: $\mathsf{F}(\mathsf{x};\mathsf{y};\mathsf{l};\mathsf{r}) = \tilde{\mathsf{F}}(\mathsf{x};\mathsf{y};\mathsf{l};\mathsf{r}\mathsf{T}(\mathsf{y}))$:

Additionally, our analysis focusses on propriatory resources such as land or managerial capital. Yet firms may also use some generic capital good. Assume this general captial can be obtained in the world market at unit price i; and a firm that produces with k units of such capital achieves output $\tilde{F}(x; y; l; r; k)$: Then the relevant production function for our analysis is $F(x; y; l; r) = \max_k \tilde{F}(x; y; l; r; k) - ik$. That is, the production function we analyze is the induced production after optimal decisions on generic capital are made. We return to this extension in Section 4.

Finally, the search literature has emphasized the role matching frictions. Models like Caribaldi and Moen (forthcoming) assume that a firm posts V vacancies at cost cV, and when l workers try to get a job at these vacancies, only $lm(l=V)$ of them are hired while the remainder stay unemployed. Here m captures the matching function, and this framework allows us to study frictional unemployement with large firms and two-sided heterogeneity. We show in the competitive search setting in Section 3: If the firm produce $\tilde{F}(x; y; h; r)$ when it hires h workers, then the induced production function that we use for our general characterizations is $F(x; y; l; r) = \max_V \tilde{F}(x; y; lm(l=V); r) - cV$; i.e., the production that arises after optimal vacancy choices have been made.

3 Assortative Matching

Models or assortative matching are in general difficult to characterize. Therefore, the literature has tried to identify conditions under which sorting is assortative. These conditions help our understanding of the underlying driving sources of sorting. In a setting like this where the welfare theorems hold, such conditions uncover the efficiency reasons behind the sorting pattners. And if the appropriate conditions are fullfilled, they substantially reduce the complecity of the assignment problem and allow further characterization of the equilibrium. In this section we derive necessary and sufficient conditions for assortative matching and characterize the assortative equilibrium.

Condition for Assortative Matching: Assume that the equilibrium is assortative, supported by some differentiable assignment function (x): Consider any $(x; \ (x); \ )$ in the support of the equilibrium allocation with $\ > 0$; i.e., with positive amount of hiring. By (2) this means that $(x; \ )$ are maximizers of the following problem for a firm of type $y = \ (x)$:

$$\max_{x,\theta} f(x; y; \ ) - \ w(x):$$

Assortative matching means that each firm only hires one type, and this problem can be understood as the problem of a firm that could choose any other worker type at any other quantity. The first order conditions for optimality are

$$f_\theta(x;\ \ (x);\ \ (x)) - w(x) = 0 \tag{3}$$

$$f_x(x;\ \ (x);\ \ (x)) - \ (x)w'(x) = 0; \tag{4}$$

where (x) and $l(x)$ are the equilibrium values. The second order condition of this problem requires the Hessian to be negative definite:

$$Hess = \begin{pmatrix} f_{\theta\theta} & f_{x\theta} - w'(x) \\ f_{x\theta} - w'(x) & f_{xx} - \ w''(x) \end{pmatrix}:$$

This requires $f_{\theta\theta}$ to be negative and the determinant $|Hess|$ to be positive, or

$$f_{\theta\theta}[f_{xx} - \ w''(x)] - (f_{x\theta} - w'(x))^2 \geq 0: \tag{5}$$

We can differentiate (3) and (4) with respect to the worker type to get

$$f_{x\theta} - w'(x) = - \ '(x)f_{y\theta} - \ '(x)f_{ll} \tag{6}$$

$$f_{xx} - \ (x)w''(x) = - \ '(x)f_{xy} - \ '(x)\left[f_{x\theta} - w'(x)\right]: \tag{7}$$

In the following three lines we successively substitute (6), (8) and then (4) into optimality condition (5):

$$\begin{aligned} - \ '(x)f_{\theta\theta}f_{xy} - \left[\ '(x)f_{\theta\theta} + f_{x\theta} - w'(x)\right]\left[f_{x\theta} - w'(x)\right] &\geq 0 \\ - \ '(x)f_{\theta\theta}f_{xy} + \ '(x)f_{y\theta}\left[f_{x\theta} - w'(x)\right] &\geq 0 \\ - \ '(x)[f_{\theta\theta}f_{xy} - f_{y\theta}f_{x\theta} + f_{y\theta}f_x= \ ] &\geq 0 \end{aligned}$$

For strictly positive assortative matching ($'(x) > 0$) it has to hold the the term in square brackets is negative, for strictly negative assortative matching the term in square brackets need to be positive. Focussing on positive assortative matching, and using the relationsship in (4), we obtain the condition:

$$f_{\theta\theta}f_{xy} - f_{y\theta}f_{x\theta} + f_{y\theta}f_x= \ \leq 0: \tag{8}$$

It turns out that this condition can more conveniently be summarized in terms of the original function $F(x;y;r;s)$; for which we know that $F(x;y;\ ;1) = f(x;y;\ )$: The following relationsships will also prove useful. Homogeneity of F implies that $-F_{34} = \ F_{33}$. Since F is constant returns, so is F_1.[3] A standard implication of constant returns it then $F_1(x;y;\ ;1) = \ F_{13} + F_{14}$: We can now rewrite (8) in terms of $F(x;y;\ ;1)$ and rearrange to obtain the following cross-margin-complementarity condition:

$$\begin{aligned} & F_{33}F_{12} - F_{23}\left[F_{13} - F_1 = \ \right] \leq 0 \\ \Leftrightarrow\ \ & F_{33}F_{12} + F_{23}F_{14} = \ \leq 0 \\ \Leftrightarrow\ \ & F_{12}F_{34} \geq F_{23}F_{14} \end{aligned}$$

So the condition depends on the cross-partials in each dimension, relative to the cross-partials across the two dimensions. Only if the within-complementarities in extensive and intensive deminsion on the left hand side exceed the between-complementarities from extensive to intensive margin on the right hand side does positive assortative matching arise. We sum this finding up in the following proposition:

Proposition 1 *A necessary cross-margin-complementarity condition for positive assortative matching is*

$$F_{12}F_{34} \geq F_{23}F_{14} \tag{9}$$

along the equilibrium path. The opposite inequality is necessary for negative assortative matching. If this condition holds at all $(x;y;l;r)$ *it is also sufficient to ensure existence of an assorted equilibrium.*

Proof. The proof of necessity is provided above. Since the condition ensures local concavity around the equilibrium path, it is sufficient to ensure that local deviations are not profitable. The proof for global sufficiency is yet to be done. ■

Interpreting this condition is relatively straightforward: On the left-hand side, a high cross-partial on the quality dimensions (F_{12}) means that higher types have ceteris paribus a higher marginal return for matching with higher types on the other side. This is reinforced by a higher cross-partial on the quality dimension, even though under constant returns to scale this can be viewed as a normalization. More importantly is the interpretation of the terms on the right-hand side. Consider the cross-partial F_{23}: If this is high, it means that we are in a setting where higher firms have a higher marginal valuation for the quantity of workers. That is, better firms value the number of "bodies" that work for them

[3] It holds that ( ) = (1) so differentiation implies that $_1($) = $_1($ 1).

especially high. It turns out that high quality workers are expensive, since they command higher wages. If the firms are predominantly interested in the number of bodies, they rather higher the less able workers but lots of them. These workers are cheaper, and overall profits go up.

The importance of the right hand side relies on the ability to substitute additional workers to make up for their lower quality. The following discussion reveals that as the elasticity of substition on the quantity dimension goes to zero in a way that agents can only be matched into pairs, the importance of the right hand side vanishes. It also dicusses other settings from the literature that arise as special cases.

Corollary 1 *Condition (9) includes the following special cases:*

Efficiency units of labor. *A particularly common assumption in the literature is the case of efficiency units of labor, where the output remains unchanged as long as the multiplicative term* xl_x *remains unchanged. In such a case workers of one type are completely replaceable by workers of half the skills as long as there are twice as many of them. Sorting is then essentially arbitrary: Each firm cares only about the right total amount of efficiency units, but not whether they are obtain by few high-type workers or many low-type workers. Our setup captures efficiency units of labor under production function* $\mathsf{f}(\mathsf{x}; \mathsf{y}; \mathsf{l}) = \tilde{\mathsf{f}}(\mathsf{y}; \mathsf{xl})$: *Taking cross-partials immediately reveals that we always obtain* $\mathsf{F}_{12}\mathsf{F}_{34} = \mathsf{F}_{23}\mathsf{F}_{14}$ *in this case.*

Multiplicative separability of the quality and quantity dimensions. *A particularly tractable case arises under multiplicative separability of the form* $\mathsf{F}(\mathsf{x}; \mathsf{y}; \mathsf{l}; \mathsf{r}) = \mathsf{A}(\mathsf{x}; \mathsf{y})\mathsf{B}(\mathsf{l}; \mathsf{r})$. *In this case the condition (9) for positive assortative matching can be written as* $[\mathsf{AA}_{12}=(\mathsf{A}_1\mathsf{A}_2)][\mathsf{BB}_{12}=(\mathsf{B}_1\mathsf{B}_2)] \geq 1$: *If* B *has constant elasticity of substitution* "; *we obtain an even simpler condition* $\mathsf{AA}_{12}=(\mathsf{A}_1\mathsf{A}_2) \geq$ ":[4]

Becker's one-on-one matching model as a limit case. *Consider some output process* $\mathsf{F}(\mathsf{x}; \mathsf{y}; \mathsf{l}; \mathsf{r})$. *In the spirit of most of the sorting literaterature, we can now consider the restricted variant where only "paired" inputs can operate: every worker needs exactly one unit of resource and any resources needs exactly one worker, and the remainder are idle. In this setting the true production is given by* $\mathsf{F}(\mathsf{x}; \mathsf{y}; \min\{\mathsf{l}; \mathsf{r}\}; \min\{\mathsf{r}; \mathsf{l}\}) = \mathsf{F}(\mathsf{x}; \mathsf{y}; 1; 1)\min\{\mathsf{l}; \mathsf{r}\}$; *where the equality follows from constant returns to scale. This corresponds to the multiplicatively separable setup we discussed in the previous point. While our framework is build around the idea that more resources or more labor inputs improve production,*

[4] If is in the unit interval, this condition is equivalent to root-supermodularity, i.e., it is equivalent to $\sqrt[n]{\ (\ \ )}$ being supermodular with $= (1 - \ )^{-1}$ as shown by Eeckhout and Kircher (2010) in a pairwise matching framework with directed search frictions. If 1 this requires conditions on ( ) that are stronger than log-supermodularity.

this Liontieff setup on the quantity dimension is exactly the limit case of a CES function with zero elasticity ($" \to 0$)*: From the previous point we therefore know that sorting arises in this limit if* $F_{12} \geq 0$; *which is exactly the condition in Becker (1973).*

Sattinger's span of control problem as a limit case. *One of the few contributions that provides clear conditions for sorting in a many-to-one matching model is presented in Sattinger (1975). His model can be described as follows. The manager of each firms has one unit of time available. Each employed worker* x *produces one unit of output, but requires* $t(x;y)$ *units of supervision-time from a manager type* y*, where higher types need less time. This means that the manager can supervise* $g(x;y) = t(x;y)^{-1}$ *workers per unit of time. Devoting* r *units of type allows him to supervise no more than* $rg(x;y)$ *workers, and since the rest remain idle the production is with* l *workers is limited at* $F(x;y;l;r) = \min\{rg(x;y);l\}$: *Our model allows for more flexibility in the substitution between inputs, but a CES extension that takes* $rg(x;y)$ *and* l *as inputs again has the previous Liontieff specification as the inelastic limit*:[5] *Inspecting (9) and taking the inelastic limit reveals that positive sorting arises only if* $g(x;y)$ *is log-superodular. This exactly recovers the condition found by Sattinger.*

Spatial Sorting Within the Mono-centric City. *The canonical model of the mono-centric city can explain how citizens locate across different locations, however there is no spatial sorting. All agents are identical and in equilibrium they are indifferent between living in the center or in the periphery by trading off commuting time for housing space and prices.*[6] *Let there be a continuum of locations* y*, each with housing stock* $r(y)$*. Let* $y \in [0;1]$*, where* y *is the center and* y *is the inverse of a measure of the distance from the center. Agents with budget* x *have preferences over consumption* c *and housing* h *represented by a quasi-linear utility function* $u(x;y) = c + v(h)$*. With consumption the numeraire good and* $p_h(y)$ *the price per unit of housing in location* y*, the budget constraint is* $c + p_h(y)h = xg(y)$*, where* $g(y)$ *is an increasing function of* y *with* $g(y) = 1$*. The closer to the center, the less time is spent on commuting and the more time is spent on productive work. Then we can write the individual citizen* x*'s optimization problem as* $xg(y) + v(h) - p_h(y)h$*. The total supply of housing in location* y *is* r *and as a result,* $l \cdot h = r$*. Net of the transfers, the aggregate surplus for all* l *citizens is given by* $F(x;y;l;r) = xg(y)l + v\left(\frac{r}{l}\right)l$*. It is easily verified that* $F_{12} = g'(y)l; F_{34} = -\frac{r}{l}v''\left(\frac{r}{l}\right); F_{14} = 0$ *so that if* $v(\cdot)$ *is concave there is positive assortative matching of the high income earners into the center and*

[5] The function ( ) = ([( )]$^{(\varepsilon-1)/\varepsilon}$ + $^{(\varepsilon-1)/\varepsilon}$)$^{\varepsilon/(\varepsilon-1)}$ approaches min{ ( ) } as $\to 0$

[6] Also Lucas and Rossi-Hansberg (2002) model the location of identical citizens but their model incorporates productive as well as residential land use. Though agents are identical, they earn different wages in different locations. The paper proves existence of a competitive equilibrium in this generalized location model which endogenously can generate multiple business centers.

the low income earners in the periphery. A similar functional form is used in Van Nieuwerburgh and Weill (2010) to consider differences between cities (rather than within the city) where the term $xg(y)$ *is replaced my a more agnostic worker-output* $u(x; y)$ *depending on worker skill* x *and city type* y: *Again, sorting is driven by the cross-partial* x *and* y *because* $F_{14} = 0$:

The sorting conditions of the previous proposition establishe a positive relation between the productivity of the firm and the skills of the workers that it hires. It does not directly establish how many workers a given firm hires, and therefore it does not directly determine even under assortative matching who matches with whom.

Efficiency: One particular beauty of earlier work on one-on-one matching such as Becker (1973) is due to the fact that their condition for assortative matching can be understood by a simple efficiency consideration. If the production function is strictly supermodular but some agents are matched negatively assortative, a simple re-arrangement such that both high types and both low types are paired together increases efficiency. These efficiency gains induce assortative matching in the market game. Since our setisfies the first welfare theorem and we have quasi-linear preferences, it is easy to see that our condition has to relate to the efficiency of the underlying assignment problem. We highlight the connection by providing an analogue to the efficiency result in the previous literature. We present the result in terms of mass points of agents that match negatively assortative to make it as comparable to the previous literature as possible. Clearly it carries over to distributions with density when the efficiency increases at all the various points of negative sorting are added up.

Consider type distributions that allow for mass points, and therefore the distribution of resources allows for mass points. A feasible distribution $\mathcal{R}$ generates market output $O(\mathcal{R}) = \int F(x; y; \ ; 1) d\mathcal{R}$. The following result states that if the production function F fulfills strict cross-margin-supermodularity, the output is never maximized when $\mathcal{R}$ matches a positive measure of agents into combinations $(x_1; y_1)$ and $(x_2; y_2)$ that are negatively assortated.

Proposition 2 *Assume* $F_{12}F_{34} > F_{23}F_{14}$ *at all* $(x; y; l; r)$: *Assume a feasible resource allocation* $\mathcal{R}$ *matches a measure* $r_i > 0$ *of resources at combination* $(x_i; y_i; \ _i)$ *for* $i \in \{1; 2\}$ *where* $x_1 > x_2$ *but* $y_1 < y_2$: *Then there exists another feasible resource allocation* $\mathcal{R}'$ *that achieves higher output:* $O(\mathcal{R}') > O(\mathcal{R})$:

Proof. See Appendix. ■

In fact, an improvement in output can be achieved by positively assortative rematching only agents of types $x_1; x_2; y_1$ and y_2, while retaining the matching among all other agents. The main difficulty of the proof is to assign the right fraction of agents together, which is no longer necessarily one-to-one. This is indeed the key insight in this theory, which exploits the fact that output can be improved by improving the factor intensity, not just the matching pattern. An additional difficulty is that x_1 and x_2 are not necessarily close to each other, and neither are y_1 and y_2: While this is solved in one-to-one matching models by integrating the marginal gains over the cross-partial, this is more difficult in our setting where the condition involves not just one cross-partial. The appendix deals with both problems.

Factor Intensity and Assignment in Assortative Equilibria: In contrast to models with pair-wise matching where assortativeness immediately implies who matches with whom (the best with the best, the second best with the second best, and so forth), this is not obvious in this framework as particular firms may hire more or less workers in equilibrium. The following differential equations describe the equilibrium assignment in this model.

Proposition 3 *If (9) holds, then the equilibrium assignment and factor intensity are uniquely determined by the system of differential equations:*

$$'(x) = \frac{h_w(x)}{(x)h_f(x)} \quad ; \quad '(x) = \frac{1}{f_{\theta\theta}}\left[\frac{1}{\ } f_x - \frac{h_w}{h_f} f_{y\theta} - f_{x\theta}\right] \tag{10}$$

Proof. The market clearing condition implies:

$$H_w(\bar{x}) - H_w(x) = \int_{\mu(x)}^{\bar{y}} (\tilde{x})h_f(\tilde{x})dx$$

Differentiating with respect to x gives the first differential equation in (10), with initial condition $(\bar{x}) = \bar{y}$ in the case of PAM.

From the first-order condition in equation (3) we know that $f_\theta(x; \ (x); \ (x)) = w(x)$ and from (4) we know that $w' = f_x = \ $. Then from equation (6), after substituting for w' and $\ '$ we obtain:

$$\frac{f_x}{\ } = f_{x\theta} + \frac{h_w}{h_f} f_{ys} + \ ' f_{\theta\theta};$$

which is equivalent to the second equation in (**??**). The initial condition for this differential equation obtains from running down the allocation from the top to the bottom and where the boundary condition holds either when the lowest type is attained or when the number of searchers goes to zero. An equilibrium allocation simultaneously solves the differential equation for $\ '$ and $\ '$ with the respective boundary conditions. ■

Notice that the an increasing …rm size means that the interaction between quality and quantity is at least sometimes positive. Otherwise it is better to concentrate one's resources on fewer and fewer workers as the workers become more productive, which can be observed in some specialized industries where highly specialized groups of workers get equipped with much captial.

Corollary 2 For a general technology, a necessary condition for the …rm size to be increasing is that at least one of f_{xs} or f_{ys} be positive.

Proof. It is immediate from inspection of the expression of $s^0(x)$ that if both f_{xs} and f_{ys} are negative, s^0 is negative as well since $f_{ss} < 0$ and $f_x > 0$. ■

A useful example to illustrate the nature of the assignment function and of factor intensity is again the case where the quantity dimension is multiplicatively separable and CES, as discussed in Corollary 1:

$$\text{Example: } F(x; y; l; r) = A(x; y)(\quad l \quad + (1 \quad) r \quad)^{1=} ; \tag{11}$$

with elasticity of substitution " = (1) ^{1} 2 (0; 1) to ensure our concavity properties on the production function, and 2 (0; 1): As discussed earlier, the condition for positive assortative matching reduces to $A_{12}A \quad A_1A_2=(1 \quad)$, the strength of which depends on : This illustrates the importance of the subsitutability of production factors on the intensive margin. When is close to 1 so that production factors are very substitutable, this condition is very di¢ cult to ful…ll. If the inequality above is ful…lled, we can apply (10) and obtain after some re-organization the assignment equations

$$ {}^{0}(x) = \quad (x) \quad ^{1} \quad ; \qquad {}^{0}(x) = \frac{(1 \quad) A_2(x; \quad (x)) \qquad A_1(x; \quad (x)) \quad ^{1}}{A(x; \quad (x))[1 + \quad][1 \quad]} : \tag{12}$$

Symmetry is the …rst observation that follows from inspection of these equations. Note that under = 1=2 and A(x; y) = A(y; x) the problem is exactly symmetric in …rms and workers. Inspecting (12) reveals that (x) = 1 and (x) = x solves this system, and in fact is also ful…lls the boundary conditions that neither workers nor …rms remain unmatched while their neighbors still earn positive returns (since workers and …rms exactly are matching one-on-one).

Asymmetries arise when for example the resources of …rms become more important than those of workers. This is the case when < 1=2. It is easy to see that for the benchmark assignment (x) = x is not longer sustainable; rather there is a shift of resources towards the more productive …rms (^{0} > 0): The opposite arises under > 1=2 where workers are more important and better worker types are endowed with more of the firm's resources, leading to reductions in firms size to equip every workers with enough endowment.

The limiting case of one-on-one matching can again be envisioned as the limit where factors are difficult to substitute ($" \to 0; \quad \to -\infty$): As discussed in Corollary 1, the condition for sorting in this case reduces the supermodularity in the quantity diemensions, as in Becker (1973). In terms of assignment and factor intensity, inspection of (12) reveals for this limit case that $'(\mathsf{x}) = 0$; which implies that $(\mathsf{x}) = 1$ and $'(\mathsf{x}) = 1$; which again implies $(\mathsf{x}) = \mathsf{x}$: Therefore, the factor intensity and assignment also converges exactly the case of Becker (1973).

The importance of the more general conditions in (9), (10) and (**??**) is exactly to highlight the relevant sorting and assignment conditions when substitution between firm and worker inputs is not impossible.

Frictions and Involuntary Unemployment

Involuntary unemployment is not present in the model outlined so far. The model makes strong predictions on the wages that workers earn and their factor intensities, it makes no predictions on their probability of being employmed. For some applications this might be rather limiting.

One simple frictional interpretation of our framework that takes partially care of this is the following: assume each firm has only a single job, but the number l of workers that it attracts constitute potential applicants who have to go through a matching function with the standard feature that the probability of filling the job goes up with the number of applicants. Since expected output is the product of the matching probability and the output produced when hiring, it is muliplicatively seperable and the sorting condition coincides with that in the second example in Corollary 1 (Eeckhout and Kircher (2010)).

The main drawback of this approach is that it does not capture the feature of true multi-worker firms: decreasing returns in production and actual choices of the number of jobs that are posted. The sorting framework that we laid out in the previous section is well-suited to capture true multi-worker firms with decreasing returns in production. In this section we embedd the previous setup in a costly recruiting and search process that has been used in other settings to capture the hiring behavior of large firms, albeit related work did not handle the two-sided heterogeneity. We will be able to derive predictions not only on the expected wages but also about the unemployment rate of workers of different skills. In particular, there exists a simple positive link between the worker's wages and their employment prospects. The following setup builds on the competitive search literature (e.g., Peters 1991; Acemoglu and Shimer 1999; Burdett, Shi and Wright 2001; Shi 2001; Shimer 2005; Eeckhout and Kircher 2009; Guerrieri, Shimer and Wright 2010) and its extensions to the analysis of multiworker firms (Menzio and Moen 2010; Garibaldi and Moen forthcoming; Kaas and Kircher 2011). We borrow the standard assumptions made in multi-worker firm models with search frictions, with the inoovation being the heterogeneity of both workers and firms.

Consider a situation where the workers are unemployed and can only hired by firms via a frictional hiring process. As part of this process, each firm decides how many vancancies v_x to post for each worker type x that it wants to hire. Posting v_x vacancies has a linear cost cv_x. It also decides to post wage ω_x for this worker type. Observing all vacancy postings, workers decide where to search for a job. Let q_x denote the "queue" of workers searching for a particular wage offer, defined as the number of workers per vacancy. Frictions in the hiring process make it impossible to fill a position for sure. Rather, the probability of filling a vacancy is a function of the number of workers queueing for this vacancy, denoted by $m(q_x)$, which is assumed to be strictly increasing and strictly concave.[7] Since there are q_x workers queueing per vacancy, the workers' job-finding rate for these workers is $m(q_x)/q_x$. The job finding rate is assumed to be strictly decreasing in the number of workers q_x queueing per vacancy. Firms can attract workers to their vacancies as long as these workers get in expectation their equilibrium utility, meaning that q_x adjusts depending on ω_x to satisfy: $\omega_x m(q_x)/q_x = w(x)$. Note the difference between the wage ω_x which is paid when a worker is actually hired, and the expected wage $w(x)$ of a queueing worker who does not yet know whether he will be hired or not. In equilibrium the firm takes the latter as given because this is the utility that workers can ensure themselves by searching for a job at other firms, while the former is the firm's choice variable with which it can affect how many workers will queue for its jobs. Therefore, a firm maximizes instead of (1) the new problem

$$\max_{r_x,\omega_x,v_x} \int [F(x, y, l_x, r_x) - l_x \omega_x - v_x c]\, dx \qquad (13)$$
$$\text{s.t. } l_x = v_x m(q_x); \quad \text{and} \quad \omega_x m(q_x)/q_x = w(x)$$

and r_x integrates to unity. The first line simply takes into account that the firm has to pay the vacancy-creation cost, and that the number of hires depends on the amount of hiring per vacancy which is in

[7] Careful elaborations how this queueing problem in a finite economy translates into matching probabilities as the population is exanded is given e.g. in Peters (1991) and Burdett, Shi and Wright (2001). It is based on the idea that workers approach vacancies unevenly due to coordination problems, which leads to excess applicants at some vacancies and to few vacancies at others.

turn related to the wage that it offers. There are two equivalent representations of this problem that substantially simplify the analysis. It can easily be verified that problem (13) is mathematically equivalent to both of the following two-step problems:

1. Let $\mathsf{G}(\mathsf{x};\mathsf{y};\mathsf{s};\mathsf{r}) = \max_v [\mathsf{F}(\mathsf{x};\mathsf{y};\mathsf{v}\mathsf{m}(\mathsf{s}{=}\mathsf{v});\mathsf{r}) - \mathsf{vc}]$; and solve $\max_{s_x,r_x} \int [\mathsf{G}(\mathsf{x};\mathsf{y};\mathsf{s}_x;\mathsf{r}_x) - \mathsf{w}(\mathsf{x})\mathsf{s}_x]\mathsf{dx}$ where r_x integrates to unity.

2. Let $\mathsf{C}(\mathsf{l};\mathsf{x}) = \min_{v,q}[\mathsf{cv} + \mathsf{vqw}(\mathsf{x})]$ s.t. $\mathsf{l} = \mathsf{vm}(\mathsf{q})$; and solve $\max_{s_x,r_x} \int [\mathsf{F}(\mathsf{x};\mathsf{y};\mathsf{l}_x;\mathsf{r}_x) - \mathsf{C}(\mathsf{l}_x;\mathsf{x})]\mathsf{dx}$ where r_x integrates to unity.

In the first equivalent formulation, the firm attracts "searchers" s_x, which queue up to get jobs at this firm. In order to entice them to do this, it has to offer wage $\mathsf{w}(\mathsf{x})$ in expectation to them whether or not they actually get hired. The definition of G then reflects the fact that the firm can still decide how many possible vacancies to create for these workers. If the firm creates more vacancies, searchers have an easier time finding a vacancy suitable to them, and this increases the amount of actual labor that is employed within the firm. In the second formulation the firm the output minus the costs of hiring the desired amount of labor. The costs include both the vacancy-creation costs as well as the wage costs, where again the expected wage has to be paid to all workers that are queueing for the jobs.

This has two direct consequences:

Problem 1:.It has the beauty that G is fully determined by the primitives, and can be directly integrated into the framework we laid out in Section 2 (where now G replaces F): The firm looks as if it hires "searchers" which have to be paid their expected wage. Applying the machinery from the previous section allows us to assess whether sorting is assortative, and what the expected wages $\mathsf{w}(\mathsf{x})$ are that are paid in equilibrium. We take this formulation embedded in the equilibrium definition of the previous section as the definition of a competitive search equilibrium with large firms.[8]

Problem 2: It then relates the expected wages $\mathsf{w}(\mathsf{x})$ that were determined in the previous problem into job finding probabilties of the searchers. Substituting the constraint in Problem 2 into the objective function and taking first order conditions yields the main characterization of this section. It can best be expressed by writing the elasticity of the matching probability as $(\mathsf{q}) := \mathsf{qm}'(\mathsf{q}){=}\mathsf{m}(\mathsf{q})$ and by denoting the queue length that solves the minimization problem by $\mathsf{q}(\mathsf{x})$: We then obtain

$$\mathsf{w}(\mathsf{x})\mathsf{q}(\mathsf{x}) = \frac{(\mathsf{q}(\mathsf{x}))}{1 - \ (\mathsf{q}(\mathsf{x}))}\mathsf{c} \tag{14}$$

[8] The same mathematical structure arises (after rearranging) when we start with an equilibrium definition in the natural way that is usually used in the competitive search literature, where firms compete in actual wages and not in terms of expected wage payments.

The right hand side is related to the well-known Hosios condition (Hosios, 1990), which showed that efficient vacancy creation is related to the elasticity of the matching function. The condition becomes particularly tractable in commonly used settings in which the elasticity is constant. In this case the queue length that different workers face is inverse proportional to the expected utility that they obtain in equilibrium. Since better workers obtain higher expected utility $w(x)$ as determined in Problem 1 (otherwise a firm could higher better workers at equal cost), they face proportionally lower competition for each job and correspondingly higher job finding probabilities. This arises because the opportunity costs of having high skilled workers unsuccessfully queue for employment is higher, and therefore firms are more willing to create enough vacancies to enable most of these applicants to actually get hired for the job. The logic applies even if the elasticity is not constant:

Proposition 4 *In the competitive search equilibrium with large firms, higher skilled workers have face lower unemployment rates.*

Proof. The term $\ (q)=[q(1-\ (q))] = m'(q)=[m(q)-qm'(q)]$: This term is strictly decreasing in q; since the numerator is strictly decreasing and the denominator is strictly increasing in q: Since $w(x)$ is increasing in x in any equilibrium, implicit differentiation of (14) implies that $q(x)$ is decreasing, which in turn implies that the chances of finding employment are increasing in x: ■

Interestingly, this implies that under positive assortative matching the firm-size can be increasing in firm type even though the number of workers that apply for jobs is decreasing. This can be seens mathematically as follows. The amount of labor that is actually hired, $l(x)$; relates to the actual number of searchers and their queue per vacancy as $l(x) = s(x)m(q(x))=q(x)$; implying:

$$l'(x) = s'\frac{m}{q} + s\frac{m'q - m}{q^2}q':$$

The change in the number of searchers (s') is determined by (10) under appropriate change of variables (and f replaced by s and g): Even if the number of workers that search for employment at better firms is not increasing, the number of hires might still be increasing because the second term is strictly positive. The reason is that is high ability firms put more resources into creating jobs for their high-skilled applicants. (XXX JAN: DO WE KNOW ANYTHING ABOUT THIS?)

4 Additional Examples

Our main example was the assignment of workers to firms, which can be viewed as the assignment of resources by firms to particular worker types. We expanded this leading example to the case of unemployment. The following gives extensions and other interpretations of our setup.

Example 1: Additional Capital Inputs. Consider a production process that not only takes as inputs the amount of labor and of proprietary firm resources, and creates output $\hat{F}(x, y, l, r, k)$. The generic capital k that can be bought on the world market at price i.[9] Optimal use of resources requires $F(x, y, l, r) = \max_k \left[\hat{F}(x, y, l, r, k) - ik\right]$, where F is constant returns in its last two arguments if $\hat{F}$ is constant returns in its last three arguments. Rewriting the cross-margin-complementarity condition (9) in terms of the new primitive yields the following condition for positive assortative matching: $\hat{F}_{12}\hat{F}_{34}\hat{F}_{55} - \hat{F}_{12}\hat{F}_{35}\hat{F}_{45} - \hat{F}_{15}\hat{F}_{25}\hat{F}_{34} \geq \hat{F}_{14}\hat{F}_{23}\hat{F}_{55} - \hat{F}_{14}\hat{F}_{25}\hat{F}_{35} - \hat{F}_{15}\hat{F}_{23}\hat{F}_{45}$.

Example 2: Monopolistic Competition. In the previous sections, we analyzed the case where the firm's output is converted one-for-one into agents utility. Therefore, there are no consequences on the final output price of the good, which is normalized to one. An often used assumption in the trade literature concerns consumer preferences pioneered by Dixit and Stiglitz (1977) which are CES with elasticity of substitution $\ \in (0, 1)$ among the goods produced by different firms. For these preferences it is well-known that a firm that produces output $\tilde{f}$ has achieves a sales revenues $\ \tilde{f}^{\rho}$, where is an equilibrium outcome that is viewed as constant from the perspective of the individual firm.[10] The difficulty in this setup is that, despite the fact that output is constant returns to scale in employment and firm resources, the revenue of the firm has decreasing returns to scale. Therefore, we cannot directly apply (9). But we can conjecture that there is assortative matching so that the firm employs only one worker type, in which case revenues are $f(x, y, l) = \ \tilde{f}(x, y, l)^{\rho}$, and we can apply (8) directly. Rearranging and using $\tilde{F}(x, y, l, r) = r\tilde{f}(x, y, l/r)$ we get the condition for positive assortative matching

$$\left[\tilde{F}_{12} + (1 - \ )(\tilde{F})\frac{\partial^2 \ln \tilde{F}}{\partial x \partial y}\right]\left[\tilde{F}_{34} - (1 - \ )l\tilde{F}\frac{\partial^2 \ln \tilde{F}}{\partial l^2}\right]$$
$$\geq \left[\tilde{F}_{23} + (1 - \ )\tilde{F}\frac{\partial^2 \ln \tilde{F}}{\partial y \partial l}\right]\left[\tilde{F}_{14} + (1 - \ )\left(l\tilde{F}_{13} - l\tilde{F}\frac{\partial^2 \ln \tilde{F}}{\partial x \partial r}\right)\right].$$

[9] This section expands on the short exposition in Footnote **??**.

[10] The underlying form for the utility function is $= {}_0^{1-\mu}\left(\int ( )^{\rho} \right)^{\mu/\rho}$ where $_0$ is a numeraire good and ( ) is the amount of consumption of the good of producer Then one obtains $= (p \ )^{1-\rho} \ {}^{\rho}$ where is the aggregatve income, $_y$ denotes the price achieved by firm through its equilibrium quantity, and $= \left(\int \ {}_y^{\rho/(1-\rho)}\right)^{\rho/(1-\rho)}$ represents the aggregate price index

Several points are note-worthy. First, the condition is independent of ; and therefore can be checked before this term is computed as an outcome of the market interaction. Furthermore, for elastic preferences ($\to 1$) the condition reduces to our original condition (9). In gereral, the condition relies not only on supermodularities in the production function, but also on log-supermodularities. This should not be surprising. Even in the standard models supermodularity is the relevant condition when the marginal consumption value of output is normalized to one (Becker 1973), while sorting when output is CES-aggregated requires log-supermodularity. If $\tilde{F}$ is multiplicatively separable between quantity and quality dimension, and the quality dimension is CES, then as the quality dimension becomes increasingly inelastic it is easy to show that the condition reduces to log-supermodularity in x and y:

Example 3: Optimal transport. Assume it costs $-r * c(x; y)$ to move a r units of waist from production site x into destination storage y; and if one attempts to move more units r into any given amount l of storage then there is some probability of damage $d(r/l)$ that each unit that is stored gets destroyed. This leads to function $F(x; y; l; r) = -rc(x; y) - \ rd(r/l)$; where represents the lost revenue because of destruction.

Example 4: Frictional matching of men and women. Assume that there are different locations where men and women can meet. These are located in different distinct locations. If r men of type x search for s women of type y; then $M(r; s) \leq \min\{r; s\}$ matches are created, where M is a standard constant returns to scale matching function. Each match is worth $A(x; y)$ for the pair. Then output is given by $F(x; y; r; s) = M(r; s)A(x; y)$: This is essentially the setup in Eeckhout and Kircher (2010). Under the standard assumption that the search literature makes on M; they find a necessarily and sufficient condition for positive assortative matching is that A is root-supermodular. In a general production environment it is not possible to reduce the analysis to conditions on the quality dimension only, and a more general look at the cross-margin-complementarity is required.

Example 5: Matching of two teams of workers. Rather than thinking about complementarities between the worker and the firm, one can equally well consider complementarities between teams of workers. One way to think about such complementarities may be the following. There is only one quality dimension Q and some distribution $H^Q(q)$ of worker skills. But production needs two teams. So if one team has quality $x \in Q$ and l team members while the other team has quality $y \in Q$ and r team members, then output is $F(x; y; l; r)$: Since the teams are drawn from the same base population, the natural requirement is symmetry of the form $F(x; y; l; r) = F(y; x; r; l)$; at least in settings where the cooperation is symmetric. Then $X = Y = Q$; and each side gets half the number of available workers $\mathsf{H}^f(\mathsf{x}) = \mathsf{H}^w(\mathsf{x}) = \mathsf{H}^Q(\mathsf{x})$=2: This provides a stylized way to analyze patterns of teams that work together.

5 Concluding Remarks

We have proposed a matching model that incorporates factor intensity and unemployment. We derive a simple condition for assortative matching and characterize the equilibrium firm size, unemployment level and unemployment by skills.

6 Appendix

Proof. Proof of Proposition 2: Strict cross-margin-supermodularity $F_{12}F_{34} > F_{14}F_{23}$ for all $(x; y; l; r)$ is by (8) equivalent to $f_{\theta\theta} f_{xy} - f_{y\theta} f_{x\theta} + f_{y\theta} f_x = \ < 0$ for all $(x; y; \ )$. Assume a feasible resource allocation $\mathcal{R}$ matches a measure $r_i > 0$ of resources at combination $(x_i; y_i; \ _i)$ for $i \in \{1; 2\}$ where $x_1 > x_2$ but $y_1 < y_2$: We will establish that output is strictly increased under a feasible variation yielding resource allocation $\mathcal{R}'$ that pairs some of the x_2 workers to some of the y_2 resources. We proceed in two steps. Step 1 has the key insight.

1. Establish the marginal benefit from assigning additional workers to some resource type:

Consider some $(x; y; \ )$ such that r resources are deployed in this match (and are paired to $\ r$ workers). For the variational argument, we are interested in the marginal benefit of pairing an additional measure r' of resources of type y' with workers of type x. The optimal output is generated by withdrawing some optimal measure $\ ' r'$ of the workers that were supposed to be working to with resource y and reassigning them to work with resource y': The joint output at $(x; y)$ and $(x; y')$ is given by

$$r f(x; y; r; \ - \ ' r' = r) + r' f(x; y'; \ '): \tag{15}$$

Optimality of $\ '$ requires according to the first order condition that $f_3(x; y; \ - \ ' r' = r) = f_3(x; y'; \ ')$; which shows that the optimal $\ '$ is itsself a function of r'. Denote $\ (y'; x; y; \ )$ the marginal increase of (15) from increasing r'; evaluated at $r' = 0$: It is given by

$$(y'; x; y; \ ) = f(x; y'; \ ') - \ ' f_3(x; y'; \ ') \tag{16}$$

$$\text{where } \ ' \text{ is determined by } \ f_3(x; y'; \ ') = f_3(x; y; \ ): \tag{17}$$

The constrained (17) reiterates the optimality of $\ '$ as a function of $x; y; \ $ and y'. The cross-partial $\ _{12}$ of the marginal benefit in (16) with respect to x and y' is strictly positive, evaluated at $y' = y$; iff

$$f_{xy} > -[\ f_{y\theta} f_{x\theta} + f_{y\theta} f_x] = [\ f_{\theta\theta}];$$

i.e., exactly when our cross-margin condition holds. Therefore, it is optimal to assign higher buyers to higher sellers locally around $(x; y)$: This is at the heart of the argument. The next step simply extends this logic to a global argument where y' might be far away from y:

2. Not PAM has strictly positive marginal benefits from matching the high types:

We started under the assumption that matching is not assortative since $x_1 > x_2$ but $y_1 < y_2$:. In particular, consider y_1 matched to x_2 at queue length $_1$ and y_2 matched to x_1 at queue $_2$; where $x_2 > x_1$ and $y_2 > y_1$. For $(x_1; y_2)$ and $(y_1; x_2)$ to be matched, optimality requires that the marginal benefit of types $y^v = y_1$ are higher when paired with x_2; while types $y^v = y_2$ yield higher benefit when paired with x_1 :

$$(y_1; x_2; y_2; \ _2) \quad \leq \quad (y_1; x_1; y_1; \ _1); \tag{18}$$

$$(y_2; x_2; y_2; \ _2) \quad \geq \quad (y_2; x_1; y_1; \ _1); \tag{19}$$

where $(\cdot; \cdot; \cdot; \cdot)$ was defined in (15). We will show that if (18) holds, then (19) cannot hold, which yields the desired contradiction. We will show this by proving that the benefit $(y'; x_1; y_1; \ _1)$ on the right hand side of (18) and (19) always remains above the benefit $(y'; x_2; y_2; \ _2)$ on the left hand side. By (18) this has to be true at $y' = y_1$; and we will show that it remains true when we move to higher y': The marginal increase of with respect to its first argument y' is given by

$$_1(y^v; x; y; \ ) = f(x; y'; \ '); \tag{20}$$

where $'$ is again determined as in (17). Assume there is some $y' \geq y_1$ such that marginal benefits are equalized, i.e., $(y'; x_2; y_2; \ _2) = (y'; x_1; y_1; \ _1)$: We have established the result when we can show that $_1(y'; x_2; y_2; \ _2) < \ _1(y'; x_1; y_1; \ _1)$.

By (20) this equivalent to showing that $f(x_2; y'; \ '_2) < f(x_1; y'; \ '_1)$, where $'_1 = \ '(y'; x_1; y_2; \ _2)$ and $'_2 = \ '(y'; x_2; y_1; \ _1)$ as in (17). To show this, define (x) for all x in resemblance of (16) by the following equality

$$f(x; y'; \ (x)) - \ (x) f_3(x; y'; \ (x)) = \ (y'; x_2; y_2; \ _2);$$

which implies $(x_2) = \ '_2$ and $(x_1) = \ '_1$ by equality of the marginal benefits at y', i.e. by $(y'; x_2; y_2; \ _2) = (y'; x_1; y_1; \ _1)$. Differentiating $f(x; y'; \ (x))$ with respect to x reveals that it is strictly increasing exactly under our strict inequality $f_{\theta\theta} f_{xy} - f_{y\theta} f_{x\theta} + f_{y\theta} f_x = \ < 0$. This in turn implies $f(x_2; y'; \ '_2) < f(x_1; y'; \ '_1)$. ■

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