Dealing with Maturity : Optimal Fiscal Policy with Long Bonds*

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Abstract

We study Ramsey optimal fiscal policy under incomplete markets in the case where the government issues only long bonds of maturity $N > 1$. We find that many features of optimal policy are sensitive to the introduction of long bonds, in particular tax variability and the long run behavior of debt. When government is indebted it is optimal to respond to an adverse shock by promising to reduce taxes in the distant future as this achieves a cut in the cost of debt. Hence, debt management concerns about the cost of debt override typical fiscal policy concerns such as tax smoothing. In the case when the government leaves bonds in the market until maturity another source of tax volatility is that optimal policy imparts N-period cycles in taxes. We formulate the equilibrium recursively applying the Lagrangean approach for recursive contracts. Even with this approach the dimension of the state vector is very large. We propose a flexible numerical method to address this issue, the "condensed PEA", which substantially reduces the required state space. This technique has a wide range of applications. To explore issues of policy coordination and commitment we propose an alternative model where monetary and fiscal authorities are independent.

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1 Introduction

As the current European sovereign debt crisis emphasizes the maturity structure of government debt is a key variable. Deciding fiscal policy independently of funding conditions in the market is a doomed concept: taxes, public spending and fiscal deficits, should take into account the funding conditions in the market for bonds. Therefore debt management should not be subservient to fiscal policy and simply be in charge of "minimizing costs of funding debt", fiscal policy and debt management should be determined jointly. Table 1 shows the average maturity of outstanding government debt for a variety of countries and displays clear differences across nations. Any theory of debt management needs to explain the costs and benefits for fiscal policy of varying the average maturity.

INSERT TABLE 1 HERE

Some recent contributions have studied the interaction between debt management and taxation policy in a Ramsey equilibrium setting. Angeletos (2002), Barro (2003), Buera and Nicolini (2004) use models of complete markets. Nosbusch (2008) explores a simplified model of incomplete markets and Lustig, Sleet and Yeltekin (2009) examine an incomplete market model with multiple maturities and nominal bonds. Faraglia, Marcet and Scott (2010) argue that optimal fiscal policy and debt management should be studied in an incomplete market setup. The current paper can be seen as a first step in this direction. We extend the work of Aiyagari, Marcet, Sargent and Seppälä (2002) on optimal fiscal policy with incomplete markets to the case when bonds mature N periods after having been issued. We describe the behavior of optimal policy with long bonds and we show how to navigate computational problems.

The equilibrium in our model shows some well known features of optimal fiscal policy under incomplete markets: the government tries to smooth taxes, taxes follow a near-martingale behavior, debt is used as a buffer stock to spread tax increases over all periods after an unexpected adverse shock is realized. We also find that if the government is indebted and an adverse shock occurs the government should promise to cut taxes in future periods, when the newly issued long bonds generate a payoff. These future tax cuts "twist" current long interest rates so as to reduce the burden of existing debt. Therefore a typical debt management concern, i.e reducing the costs of debt, overrides a typical concern of fiscal policy, namely tax smoothing. This promise to cut taxes is the reason that optimal policy is time inconsistent: if the government could renege on the promise to cut taxes it will.

A further problem when dealing with long bonds is what decision to make about outstanding debt at the end of each period. Most of the literature assumes that the government buys back each period all previously issued debt and then reissues new bonds. This assumption is innocuous in models of complete markets, but matters under incomplete markets. Furthermore, as shown in Marchesi (2004) governments rarely buy back outstanding debt before redemption. To quote the UK Debt Management Office (2003) "the UK's debt management approach is that debt once issued will not be redeemed before maturity." For this reason we also study optimal policy when the government leaves long bonds in circulation until the time of maturity. We call this the "hold to redemption" case. In this case, at any moment in time the government has a full spectrum of outstanding debt with maturity until redemption of N, $N-1$ through to 1 year. The maturity profile of government debt is therefore much more complex with long bonds and hold to redemption and this will potentially impact debt management and fiscal policy. We find that optimal tax policy is even more volatile in this case: the government promises to cut taxes permanently and there are $N-$period cycles in tax policy.

Obtaining numerical simulations is not straightforward. A first difficulty is to obtain a recursive formulation of the model, to do so we adapt the recursive contracts treatment of Aiyagari et al. (2002). A second difficulty arises because the vector of state variables is typically of dimension $2N+1$ hence it grows rapidly with maturity: many OECD countries issue thirty year bonds, both France and the UK issue fifty year bonds. Solving a non-linear dynamic model with these many state variables is not feasible.[1]

To reduce the computational complexity we propose a new method, the "condensed PEA", that reduces the dimensionality of the state vector while allowing, in principle, for arbitrary precision. We show how in the case of a twenty year bond the state space is effectively only four variables. We believe this computational method has wide applicability to other models.

The fact that the fiscal authority finds it optimal to twist interest rates to minimise funding costs raises issues about the degree of commitment and policy coordination required to actually implement such a policy. To assess this we introduce a model where the fiscal authority is separate from the monetary authority setting interest rates. In this way the "twisting" of interest rates is not possible, since the fiscal authority takes interest rates as given. This setup provides a framework to understand the role of commitment in the Ramsey policy, and in the case with buyback it reduces the dimensionality of the state vector as the usual co-state variables of optimal Ramsey policy are no longer present. We find that the second moments of the model are not highly dependent on maturity.

In a calibrated example allocations, interest rates and persistence of debt are similar across maturities and across the three models of policy considered. The main difference is the long run level of debt, as longer maturities are associated with more debt. Marcet and Scott (2009) (MS) find that incomplete markets with short bonds explain better the observed persistence of debt, but they also show that incomplete markets overshoots and displays too much persistence. We find in this paper a large improvement in the match of the persistence of debt relative to MS, although it is in part due to the use of small sample statistics.

The structure of the paper is as follows. Section 2 outlines our main model, a Ramsey model with incomplete markets and long bonds when the government buys back all outstanding debt each period. Section 2 shows some properties of the model using analytic results. Section 3 studies numerical issues, introducing the condensed PEA and describing the behavior of the model numerically. Section 4 studies the model of independent powers. Section 5 considers the case of hold to redemption whilst a final section concludes.

[1] Linearization of the policy function is undesirable. First because it turns out that non-linear terms in the policy function play a crucial role even near the steady state mean. Second because of the presence of debt limits.

2 The Model - Analytic Results

Our benchmark model is of a Ramsey policy equilibrium with perfect commitment and coordination of policy authorities in which the government buys back all existing debt each period. In Sections 4 and 5 we relax these assumptions.

The economy produces a single non-storable good with a technology

$$c_t + g_t \leq 1 - x_t, \tag{1}$$

for all t, where x_t, c_t and g_t represent leisure, private consumption and government expenditure respectively. The exogenous stochastic process g_t is the only source of uncertainty. The representative consumer has utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u\left(c_t\right) + v\left(x_t\right) \right\} \tag{2}$$

and is endowed with one unit of time that it allocates between leisure and labour and faces a proportional tax rate τ_t on labour income. The representative firm maximizes profits, both consumers and firms act competitively by taking prices and taxes as given. Consumers, firms and government have full information, i.e they observe all shocks up to the current period, and all variables dated t are chosen contingent on histories $g^t = (g_t, ..., g_0)$. All agents have rational expectations.

Agents can only borrow and lend in the form of a zero-coupon, risk-free, N-period bond so that the government budget constraint is:

$$g_t + p_{N-1,t} b_{N,t-1} = \tau_t \left(1 - x_t\right) + p_{N,t} b_{N,t} \tag{3}$$

where $b_{N,t}$ denotes the number of bonds the government issues at time t, each bond pays one unit of consumption good in N periods time with complete certainty. The price of an i-period bond at time t is p_{it}. In this section we assume that at the end of each period the government buys back the existing stock of debt and then reissues new debt of maturity N, these repurchases are reflected in the left side of the budget constraint (3). In addition the value of government debt has to remain within upper and lower limits $\underline{M}$ and $\overline{M}$ so ruling out Ponzi schmes

$$\underline{M} \leq \beta^N b_{N,t} \leq \overline{M} \tag{4}$$

The term β^N in this constraint reflects the value of the long bond at steady state, so that the limits $\underline{M}, \overline{M}$ appropriately refer to the value of debt and they are comparable across maturities.[2]

We assume after purchasing a long bond the household entertains only two possibilities: one is to resell the government bond in the secondary market in the period immediately after having purchased it, the other possibility is to hold the bond until maturity.[3] Letting $s_{N,t}$ be the sales in the secondary market the household's problem is to choose stochastic processes $\{c_t, x_t, s_{N,t}, b_{N,t}\}_{t=0}^{\infty}$ to maximize (2) subject to the sequence of budget constraints:

$$c_t + p_{N,t} b_{N,t} = \left(1 - \tau_t\right)\left(1 - x_t\right) + p_{N-1,t} s_{N,t} + b_{N,t-N} - s_{N,t-N+1}$$

[2] Obviously the actual value of debt is $p_{N,t} b_{N,t}$. We substitute $p_{N,t}$ by its steady state value β^N for simplicity, nothing much changes if the limits are in terms of $p_{N,t} b_{N,t}$.

[3] We need to introduce secondary market sales $s_{N,t}$ in order to price the repurchase price of the bond.

with prices and taxes $\{p_{N,t}, p_{N-1,t}, \tau_t\}$ taken as given. The household also faces debt limits analogous to (4), we assume for simplicity that these limits are less stringent than those faced by the government, so that in equilibrium, the household's problem always has an interior solution.

The consumer's first order conditions of optimality are given by

$$\frac{v_{x,t}}{u_{c,t}} = 1 - \tau_t \tag{5}$$

$$p_{N,t} = \frac{\beta^N E_t\left(u_{c,t+N}\right)}{u_{c,t}} \tag{6}$$

$$p_{N-1,t} = \frac{\beta^{N-1} E_t\left(u_{c,t+N-1}\right)}{u_{c,t}} \tag{7}$$

2.1 The Ramsey problem

Given the above financial assets we assume taxes are determined in a Ramsey equilibrium, that is, the government has full commitment to implement the best sequence of (possibly time inconsistent) taxes and government debt knowing equilibrium relationships between prices and allocations, and the government chooses policy that is in the best interest of consumers. Using standard arguments (5), (6) and (7) can be used to substitute for taxes and consumption, and the Ramsey equilibrium can be found by solving the following government problem

$$\max_{\{c_t, b_{N,t}\}} E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_t) + v(x_t)\} \tag{8}$$

$$\text{s.t.} \qquad \beta^{N-1} E_t\left(u_{c,t+N-1}\right) b_{N,t-1} = \left(u_{c,t} - v_{x,t}\right)\left(c_t + g_t\right) - u_{c,t} g_t + \beta^N E_t\left(u_{c,t+N}\right) b_{N,t} \tag{9}$$

and (4) for all $t = 0, 1, ...$ Implicitely x_t is just defined by (1).

To simplify the algebra we define the function $S : R^2_+ \to R$ as $S(c,g) \equiv (u_c - v_x)(c+g) - u_c g$ and we let $S_t \equiv S(c_t, g_t)$ be the "discounted" surplus of the government. The Lagrangian corresponding the government problem is

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + v(x_t) + \lambda_t \left[S_t + \beta^N u_{c,t+N} b_{N,t} - \beta^{N-1} u_{c,t+N-1} b_{N,t-1} \right] \right.$$
$$\left. + \nu_{1,t}\left(\overline{M} - \beta^N b_{N,t}\right) + \nu_{2,t}\left(\beta^N b_{N,t} - \underline{M}\right) \right\}$$

where λ_t is the Lagrange multiplier associated with the government budget constraint and $\nu_{1,t}$ and $\nu_{2,t}$ are the multipliers associated with the debt limits.

The first-order conditions for the planner's problem with respect to c_t and $b_{N,t}$ are

$$u_{c,t} - v_{x,t} + \lambda_t S_{c,t} + u_{cc,t}\left(\lambda_{t-N} - \lambda_{t-N+1}\right) b_{N,t-N} = 0 \tag{10}$$

$$E_t\left(u_{c,t+N} \lambda_{t+1}\right) = \lambda_t E_t\left(u_{c,t+N}\right) + \nu_{2,t} - \nu_{1,t} \tag{11}$$

with $\lambda_{-1} = ... = \lambda_{-N} = 0$. Note that $S_{c,t} = u_{cc,t} c_t + u_{c,t} + v_{xx,t}\left(c_t + g_t\right) - v_{x,t}$.

These FOC help characterise some features of optimal fiscal policy with long bonds. Following the discussion in Aiyagari et al. (2002) we see that in the case where debt limits are non binding (11) implies that λ_t is a risk-adjusted martingale with risk-adjustment measure $\frac{u_{c,t+N}}{E_t\left(u_{c,t+N}\right)}$, indicating that in this model the presence of the state variable λ in the policy function imparts persistence in the variables of the model. The term

$$\mathcal{D}_t = (\lambda_{t-N} - \lambda_{t-N+1})\, b_{N,t-N}$$

in (10) indicates that a feature of optimal fiscal policy will be that what happened in period $t-N$ has a special impact on today's taxes. Since the first best is given by $u_{c,t} - v_{x,t} = 0$ and zero taxes, $\mathcal{D}_t > 0$ can be thought of as introducing a higher distortion in a given period, pulling the allocations away from the first best and increasing taxes. In periods t and realizations g^t where g_{t-N+1} is very high we have that the cost of enforcing the budget constraint at $t-N+1$ was high, so λ_{t-N+1} is high, and if the government is in debt $\mathcal{D}_t < 0$ so that taxes should go down at t everything else equal. Of course this is not a tight argument, as λ_t also responds to all past shocks and λ_t also plays a role in (10), but this argument is at the core of the interest rate twisting policy we identify below. In order to build up intuition for the role of commitment and to provide a tighter argument, we now show two examples that can be solved analytically.

2.2 A model without uncertainty

Assume now that government spending is constant, $g_t = \overline{g}$ and that the government is initially in debt $b^N_{-1} > 0$. In this case the long bond completes the markets so that the only budget constraint of the government is then

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t \frac{u_{c,t}}{u_{c,0}} \widetilde{S}_t &= b_{N,-1}\; p_0^{N-1}, \text{ or} \\
\sum_{t=0}^{\infty} \beta^t\; S_t &= b_{N,-1}\; \beta^{N-1} u_{c,N-1} \qquad (12)
\end{aligned}$$

where $\widetilde{S}_t = \frac{S_t}{u_{c,t}}$ is the "non-discounted" surplus of the government. This shows that for a given set of surpluses the funding costs of initial debt can be reduced by manipulating consumption so as to achieve allocations such that $c_t < c_{_{N-1}}$ for all $t \neq N$. As long as the elasticity of consumption with respect to wages is positive, as occurs with most utility functions, this will be achieved by promising a tax cut in period $N-1$ relative to other periods, formally

$$\begin{aligned}
\tau_t &= \overline{\tau} \text{ for all } t \neq N-1 \qquad (13) \\
\overline{\tau} &> \tau_{_{N-1}}
\end{aligned}$$

This promise achieves a reduction of $u_{c,N-1}$, reducing the cost of outstanding debt. In other words, the long end of the yield curve needs to be twisted up.[4] Interestingly, even though there are no

[4] This is, of course, a version of the standard interest rate manipulation already noted by Lucas and Stokey (1983) for long bonds, except that in their model the interest rate twisting occurred in the first period.

fluctuations in the economy, (13) shows that optimal policy implies that the government desires to introduce variability in taxes, even in a model without uncertainty. In other words, optimal policy violates tax smoothing.This policy is clearly time inconsistent: if the government would be able to reneg on its promise by surprise at some period $t' > 0$, $t' < N-1$ it will then pull back the promise to cut taxes in period $N-1$ and it will promise instead a tax cut in period $t'+N-1$.

2.3 A model with uncertainty at $t = 1$

The previous subsection abstracted from uncertainty. We now introduce uncertainty into our model. In the interest of obtaining analytic results we assume uncertainty occurs only in the first period, ie g is given by[5]:

$$\begin{cases} g_t = \overline{g} & \text{for } t = 0 \text{ and } t \geq 2 \\ g_1 \sim F_g & \end{cases}$$

for some non-degenerate distribution F_g. Since future consumption and λ's are known for $t \geq 1$ the martingale condition (11) implies $u_{c,t+N}\lambda_{t+1} = \lambda_t u_{c,t+N}$ and

$$\lambda_t = \lambda_1 \quad \text{for all } t > 1 \tag{14}$$

It is clear that in the case of short bonds ($N = 1$) equilibrium implies c_t and τ_t constant for $t \geq 2$.

For the case of long bonds $N > 1$ (14) implies that the FOC with respect to consumption (10) is satisfied for

$$\mathcal{D}_t = 0 \quad \text{for } t \geq 0 \text{ and } t \neq N-1, N \tag{15}$$

$$\mathcal{D}_{N-1} = \lambda_0 b_{N,-1}\ , \quad \mathcal{D}_N = (\lambda_0 - \lambda_1)\, b_{N,0} \tag{16}$$

Hence equilibrium satisfies

$$c_t = \ \ c^*(g_1) \quad \text{for } t \geq 2 \text{ and } t \neq N, N-1 \tag{17}$$

for a certain function c^*. i.e consumption is the same in all periods $t \geq 2$ and $t \neq N, N-1$, although this level of constant consumption depends on the realization of the shock g_1. Clearly, c_{N-1}, c_N also depend on the realization of g_1.

In this model, when the shock g_1 is realized the government optimally spreads out the taxation cost of this shock over current and future periods, typically the government gets in debt in period 1 if g_1 is high, so all future taxes for $t \geq 2$ are higher and future consumption lower. This would also happen with short bonds $N = 1$, what is new with long bonds is that optimal policy introduces tax volatility, since taxes vary in periods $N-1$ and N, even though by the time the economy arrives at these periods no more shocks have occured for a long time.

[5] Formally this economy is very similar to that of Nosbusch (2008).

2.3.1 An Analytic Example

To make this argument precise consider the utility function

$$\frac{c_t^{1-\gamma_c}}{1-\gamma_c} - B\frac{(1-x_t)^{1+\gamma_l}}{1+\gamma_l} \tag{18}$$

for $\gamma_c, \gamma_l, B > 0$.

Result 1 *Assume utility (18), $N > 1$ and $b_{N,-1} > 0$.*
For a sufficiently high realization of g_1 we have

$$\begin{aligned} \tau_1 &= \tau_t \text{ for all } t > 1,\ t \neq N-1, N \\ \tau_1 &> \tau_{N-1}, \tau_N \end{aligned}$$

The inequalities are reversed if $b_{N,-1} < 0$ or if the realization of g_1 is sufficiently low.

Proof
Since $\lambda_t = \lambda_1 \quad t > 1$ the FOC of optimality for this utility function yield

$$\frac{u_{c,t}}{v_{x,t}} - \frac{B + (\gamma_l + 1)\lambda_1}{(1 + (-\gamma_c + 1)\lambda_1) B} + (\lambda_{t-N} - \lambda_{t-N+1}) \mathcal{F}_t = 0 \quad \text{for } t \geq 1$$

where $\mathcal{F}_t \equiv \frac{u_{cc,t} b_{N,t-N}}{(1+(-\gamma_c+1)\lambda_1)B}$.
Consider $t = 1$, then $\lambda_{t-N} = \lambda_{t-N+1} = 0$ so that

$$\frac{u_{c,1}}{v_{x,1}} = \frac{B + (\gamma_l + 1)\lambda_1}{(1 + (-\gamma_c + 1)\lambda_1) B} \tag{19}$$

Therefore we can write

$$\frac{u_{c,t}}{v_{x,t}} - \frac{u_{c,1}}{v_{x,1}} = (\lambda_{t-N+1} - \lambda_{t-N}) \mathcal{F}_t = 0 \quad \text{for } t \geq 1 \tag{20}$$

That $\tau_t = \tau_1$ for all $t > 1$ and $t \neq N-1, N$ follows from this equation and (17).

Now we show that $\mathcal{F}_t < 0$ for $t = N-1, N$. Since $\lambda_1, B, \gamma_l > 0$ we have that $B + (\gamma_l + 1)\lambda_1 > 0$. Since $u_{c,1}, v_{x,1} > 0$ clearly (19) implies that $(1 + (-\gamma_c + 1)\lambda_1) B > 0$. Since we consider the case of initial government debt $b_{N,-1} > 0$ this leads to $b_{N,0} > 0$ and since $u_{cc,1} < 0$ we have $\mathcal{F}_t < 0$ for $t = N-1, N$.

Since for $t = N-1$ we have $\lambda_{t-N} - \lambda_{t-N+1} = -\lambda_0 < 0$ it follows

$$\frac{u_{c,N-1}}{v_{x,N-1}} < \frac{u_{c,1}}{v_{x,1}} \Rightarrow \tau_{N-1} < \tau_t \text{ for all } t > 1,\ t \neq N-1, N.$$

Also, it is clear from (19) that λ_1 is an increasing function of g_1. Since the martingale condition implies $E_t (u_{c,N} \lambda_1) = \lambda_0 E_0 (u_{c,N})$ it has to be that for large enough g_1 we have $\lambda_1 > \lambda_0$ Therefore, for $t = N$ and if g_1 was high enough we have $\lambda_{t-N} - \lambda_{t-N+1} = \lambda_0 - \lambda_1 < 0$ so that (20) implies

$$\frac{u_{c,N}}{v_{x,N}}, \frac{u_{c,N-1}}{v_{x,N-1}} < \frac{u_{c,1}}{v_{x,1}} \Rightarrow \tau_N, \tau_{N-1} < \tau_1 \qquad \blacksquare$$

Intuitively, in period $t = N - 1$ there is a tax cut for the same reasons as in section 2.2, ie in order to reduce the funding costs on initial debt. New in this section is the tax cut (for high g_1) at $t = N$. The intuition for this is clear: when an adverse shock to spending occurs at $t = 1$ the government uses debt as a buffer stock so $b_{N,1} > b_{N,0}$, as this is the way to achieve tax smoothing by financing part of the adverse shock with higher future taxes. But since future surpluses at $t = 1$ are higher than expected as a higher interest has to be serviced, the government can lower the cost of existing debt by announcing a tax cut in period N after seeing the high realized value of g_1, since this will reduce the price $p_{N-1,0}$ of period $t = 1$ outstanding bonds $b_{N,0}$. The tax cut at $t = N$ is the stochastic analog of the tax cut described in section 2.2.

2.3.2 Contradicting Tax Smoothing

To summarize, we have proved that at the time an adverse shock to spending is realized the government has to take three actions: *i)* increase taxes permanently, *ii)* increase debt permanently, *iii)* announce a tax cut when the outstanding debt matures. Effects *i)* and *ii)* are well known in the literature of optimal taxation under incomplete markets, effect *iii)* is clearly seen in this model with long bonds since the promise is made N periods ahead. Obviously in the case of short maturity $N = 1$ of Aiyagari et al. the effect of $\mathcal{D}_1$ would be felt in deciding optimally τ_1, but this effect would be confounded with the fact that g_1 is stochastic, so effect *iii)* is harder to see in a model with short bonds.

One can interpret these results as showing that some aspects of tax policy are subordinate to debt management. In models of optimal policy the government usually desires to smooth taxes. Taxes would be constant in the model of *Result 1* if the government had access to complete markets. But we find that the government changes taxes in period $N - 1$, long after the economy has received any shock. Therefore, government forfeits tax smoothing in order to enhance a typical debt management concern such as reducing the cost of debt.

Obviously this policy is time inconsistent: if the government could unexpectedly reoptimize in period $t = 2$ given its debt $b_{N,1}$ it would renege on the promise to cut taxes in period N, instead it would promise to lower taxes in period $N + 1$.

3 Optimal Policy, Simulation Results

We now turn to the case where g_t is stochastic in all periods. As is well known analytic solutions for this type of model are infeasible, so we utilise numerical results. The objective is to compute a stochastic process $\{c_t, \lambda_t, b_{N,t}\}$ that solves the FOC of the Ramsey planner, namely (9), (10) and (11). First we obtain a recursive formulation that makes computation possible, then we describe a method for reducing the dimensionality of the state space in the computations and finally we discuss the behaviour of the economy.

3.1 Recursive Formulation

Using the recursive contract approach of Marcet and Marimon (2011) the Lagrangean can be rewritten as :

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \{ u(c_t) + v(x_t) + \lambda_t S_t + u_{c,t} (\lambda_{t-N} - \lambda_{t-N+1}) b_{N,t-N} \tag{21}$$
$$+ \nu_{1,t} \left(\overline{M} - \beta^N b_{N,t} \right) + \nu_{2,t} \left(\beta^N b_{N,t} - \underline{M} \right) \}$$

for $\lambda_{-1} = ... = \lambda_{-N} = 0$.

Assume g_t is a Markov process. Notice that only current and lagged variables appear in the terms dated t in the infinite sum of this lagrangean, furthermore, the only lagged variables appearing are $(\lambda_{t-N}, \lambda_{t-N+1}, b_{N,t-N})$. This suggests that the solution satisfies the recursive structure-

$$\begin{aligned} \begin{bmatrix} b_{N,t} \\ \lambda_t \\ c_t \end{bmatrix} &= F(g_t, \lambda_{t-1}, ..., \lambda_{t-N}, b_{N,t-1}, ..., b_{N,t-N}) \\ \lambda_{-1} &= ... = \lambda_{-N} = 0, \quad \text{given } b_{N,-1} \end{aligned}$$

for a time-invariant policy function F. A proof that this is indeed a sufficient set of state variables follows from Corollary 3.1 in Marcet and Marimon (2011).[6] This allows for a simpler recursive formulation than the promised utility approach, as the co-state variables λ do not have to be restricted to belong to the set of feasible continuation variables, which would be really complicated in this model as there are so many state variables.

The state vector in this recursive formulation has dimension $2N + 1$. It is unlikely that further reductions in the dimension can be achieved purely by theoretical results. Some authors model long bonds as perpetuities with geometrically decaying coupon payments where the rates of decay rate is chosen to mimick some properties of actual bonds (Woodford (2001), Broner, Lorenzoni and Schmulker (2007), Arellano and Ramanarayanan (2008)). Although modelling long bonds in this way allows for a recursive formulation with only 3 state varaibles actual government bonds are very much unlike this geometrically decaying payoff, as they pay a very small coupon before maturity and they provide a large payoff at maturity, much closer to the zero-coupon bond that we used. Another justification for assuming decaying payoff is that it mimics a bond portfolio with fixed shares that decay with maturity. However since our ultimate goal is precisely to study the appropriate portfolio weights this would not be an attractive motivation.

3.2 The Condensed PEA

It is at present quite easy to obtain solutions of large-dimensional dynamic stochastic models using linear approximations to the policy function F. These approximations are quite accurate

[6] In this model it is possible to reduce the state space even further by recognising that the only relevant state variables are N lags of $s_t = b_{N,t} (\lambda_t - \lambda_{t-1})$. We do not exploit this feature of the model as it is very specific to this version of the model. For example, the no buyback case of section 5 needs all state variables.

in many models of interest. But in our model the debt limits are occasionally binding for debt levels similar to those observed in the real world. When debt limits are occasionally binding the derivative of F near the debt limit may be quite different from the derivative near steady state, so a linear approximation is likely to be inaccurate. Furthermore, per our discussion in section 2.1 it is clear that what determines the effect of previous commitments on today's tax is the term $\mathcal{D}_t = (\lambda_{t-N} - \lambda_{t-N+1})b_{N,t-N}$. Therefore it is the interaction between past λ's and past b_N's that determines the size and the sign of today's tax cut, not the level of λ and b_N. A linear approximation to the policy function would fail to capture this feature of the model and it would miss a key aspects of optimal policy under full commitment.[7]

Therefore it is important to use an algorithm that can capture non-linearities. Since the state vector has dimension $2N + 1$.and bonds of maturity N =10, 30 or even 50 years are not uncommon a non-linear approach rapidly becomes intractable.

To overcome this difficulty we introduce a solution method based on the Parameterized Expectation Algorithm of den Haan and Marcet (1990). PEA is useful here because it does capture the relevant non-linearities described in Section 2.3 even if the expectations are parameterized as linear functions, since (10) and the effect of $\mathcal{D}_t$ is just imposed on the solution. Also, PEA allows for a natural space reduction method that we call "condensed PEA". This reduces the dimension of the state space effectively used while keeping the option of having an arbitrarily accurate solution.

Condensed PEA goes as follows. Denote the state vector as $X_t = (g_t, \lambda_{t-1}, ..., \lambda_{t-N}, b_{N,t-1}, ..., b_{N,t-N})$. We know that theoretically all elements of X_t are necessary in determining optimal choice variables at t. Indeed, one can come up with particularly unlikely realizations or with special initial conditions for bonds where all these variables are important in determining the solution. But it is unlikely that in the steady state distribution each and everyone of these variables plays a substantial role in determining the solution. Most likely some function of these lags will be sufficient to summarize the features from the past that need to be remembered by the government in order to take an optimal decision. In the context of PEA this can be expressed in the following way.

One of the expectations requiring approximation is

$$E_t \{u_{c,t+N}\} \tag{22}$$

appearing in (11). This expectation is a function, in principle, of all elements in X_t, but it is likely that in practice a few linear combinations of X_t may be sufficient to predict $u_{c,t+N}$. There are two reasons for this. First, the very structure of the model suggests that elements of X_t are very highly correlated with each other, suggesting that a few linear combinations of X_t have as much predictive power as the whole vector. Another way of saying this is that it is enough to project on the principal components of X_t. Other methods available for reducing the dimensionality of state vectors have emphasized this aspect. The second reason is that some of the principal components of X_t may be irrelevant in predicting $u_{c,t+N}$ in equilibrium and, therefore, they can be left out of

[7] In this particular model it might be enough to include $\mathcal{D}_t$ as a state variable instead of past λ's and past b_N's. But this discussion highlights that non-linear terms are important for optimal fiscal policy. Even though a "trick" can be found for this particular model where a linear approximation may work a different trick would be needed, or may not be available, for another model. A general that avoids having to find these tricks is to use an algorithm that can capture these non-linearities.

the approximated conditional expectation. So the goal is to include in the state vector only linear combinations of X_t that have some predictive power for $u_{c,t+N}$, the remaining linear combinations can be understood as appearing in the conditional expectation with a coefficient of zero.

More precisely, we partition the state vector into two parts: a subset of n "core" state variables $\{X_t^{core}\} \subset \{X_t\}$, where $n < 2N + 1$ is small, and an omitted subset of state variables $\{X_t^{out}\} = \{X_t\} - \{X_t^{core}\}$ of dimension $1 + 2N - n$. We first solve the model including only X_t^{core} in the parameterized expectations. If the error $\phi_{t+N} \equiv u_{c,t+N} - E_t\{u_{c,t+N}\}$ found in the solution obtained with only these core variables is unpredictable with X_t^{out} we claim the solution with core variables is the correct one. If X_t^{out} can predict this error we then find the linear combination of X_t^{out} that has the highest predictive power for ϕ_{t+N}. We add this linear combination to the set of state variables, solve the model again with this sole additional state variable, check if X^{out} can predict ϕ_{t+N} and so on.[8]

Formally, given the set of core variables we define the condensed PEA as follows.[9]

Step 1 Parameterize the expectation as

$$E_t\{u_{c,t+N}\} = (1, X_t^{core}) \cdot \beta^1 \tag{23}$$

Find values for $\beta^1 \in R^{n+1}$ that satisfy the usual PEA fixed point, denoted $\beta^{1,f}$, where the series generated by $(1, X_t^{core}) \cdot \beta^{1,f}$ causes this to be the best parameterized expectation.

This solution is of course based on a restricted set of state variables so it may be inaccurate. It is therefore necessary to check if the omission of X^{out} affects the approximate solution. The next step orthogonalizes the information in X_t^{out}, this will be helpful to arrive at a well-conditioned fixed point problem in Step 4.

Step 2 [10] Obtain a long run simulation run with the approximate solution found so far. Run a regression of each element of X_t^{out} on the core variables. Letting $X_{i,t}^{out}$ be the $i-$th element, we run the regression

$$X_{i,t}^{out} = (1, X_t^{core}) \cdot b_i^1 + X_{i,t}^{res,1}$$

$b_i^1 \in R^{2N+2-n}$. It is clear that $X^{res,1}$ adds the same information to X^{core} as X^{out} would, but $X^{res,1}$ has the advantage that it is orthogonal to X^{core} so it will be useful in finding the fixed point of Step 4..

Step 3 Using a long run simulation find $\alpha^1 \in R^{n+1}$ such that

$$\alpha^1 = \arg\min_{\alpha} \sum_{t=1}^{T} \left(u_{c,t+N} - X_t^{core} \cdot \beta^1 - X_t^{res,1} \cdot \alpha \right)^2 \tag{24}$$

[8] Obviously, in practice the elements of X^{core} should be chosen judiciously so that this initial solution is not "too" inaccurate.

[9] For convenience we describe these steps with reference only to the expectation $E_t\{u_{c,t+N}\}$. In practice the expectations $E_t\{u_{c,t+N}\lambda_{t+1}\}$ and $E_t\{u_{c,t+N-1}\}$ appearing in the FOC also need to be parameterized concurrently and the steps need to be applied jointly to all conditional expectations.

[10] This step is described assuming we are interested in the steady state distribution. Of course it could be modified to take into account transitions by instead of using long run simulations using short-run simulations that try to capture the behavior of the model during the transition.

If α^1 is close to zero the solution with only X^{core} is sufficiently accurate and we can stop here. Otherwise go to

Step 4 Apply PEA adding $X_t^{res,1} \cdot \alpha^1$ as a state variable, ie parameterizing the conditional expectation as

$$E_t \{u_{c,t+N}\} = \left(X_t^{core}, X_t^{res,1} \alpha^1 \right) \cdot \beta^2$$

where $\beta^2 \in R^{n+2}$. Find a fixed point $\beta^{2,f}$ for this parameterized expectation. Since $\beta^{1,f}$ is a fixed point, since X_t^{core} and $X_t^{res,1}$ are orthogonal and since the linear combination α^1 has high predictive power the vector

$$\underset{(n+2)\times 1}{\beta^{2,f}} = \begin{pmatrix} \beta^{1,f} \\ 1 \end{pmatrix}$$

is likely to be a good initial condition for the iterations of the fixed point. Go to Step 2 with $\left(X_t^{core}, \alpha^1 X_t^{res,1} \right)$ as only state variables in the parameterized expectation, find a new linear combination, etc.

A couple of remarks end this subsection. In the presence of many state variables it has been customary in dynamic economic models to try each state variable in order, adding state variables one by one until the next variable does not change much the solution found. For example, if many lags are needed we add the first lag, then the second lag and so on; when adding another lag changes very little the solution one claims that the solution is sufficiently accurate. But it is easy to find reasons why this argument may fail. For instance, maybe the variables further down the list (a longer lag) is the relevant one.[11] This is the case, by the way, in our model, where state variable λ_{t-N} and $b_{N,t-N}$ play a key role in determining the solution at t. Or it can be that all the remaining variables together make a difference but they do not make a difference one by one. Our method gives a chance to all these variables to make jointly a difference in the solution, therefore it is more efficient in finding relevant state variables, as Step 3 selects automatically *if* the variables left out are needed and *which* of them are to be introduced.

The whole argument in this section is made for linear conditional expectations, as in (23). Of course the same idea works for higher-order terms. In order to check the accuracy for higher order terms one can use the condensed PEA with the higher-order polynomial terms, i.e one can check if linear combinations of, say, quadratic and cubic terms of X_t have predictive power in Step 2, include these in X_t^{out} and go through Steps 2 to 4 above.

[11] For another example, incomplete market models with a large number of agents need as state variables all the moments of the distribution of agents, which is an infinite number of state variables. Usually these models are solved first by using the first moment as a state variable, if the second moment does not change much the solution it is claimed that first moments are enough. But it could be that the third or fourth moment are the relevant ones, specially since the actual distribution of wealth is so skewed.

3.3 Solving the Model with Condensed PEA

The utility function (18) was convenient for obtaining the analytic results of section 2.3. In this section we use a utility function more commonly used in DSGE models:

$$\frac{c_t^{1-\gamma_1}}{1-\gamma_1}+\eta\frac{x_t^{1-\gamma_2}}{1-\gamma_2}$$

We choose $\beta = 0.98$, $\gamma_1 = 1$ and $\gamma_2 = 2$. The choice of discount factor implies we think of a period as one year. We set η such that if the government's deficit equals zero in the non stochastic steady state agents work a fraction of leisure of 30% of the time endowment.

For the stochastic shock g we assume the following truncated AR(1) process:

$$g_t = \begin{cases} \overline{g} & \text{if } (1-\rho)\,g^* + \rho g_{t-1} + \varepsilon_t > \overline{g} \\ \underline{g} & \text{if } (1-\rho)\,g^* + \rho g_{t-1} + \varepsilon_t < \underline{g} \\ (1-\rho)\,g^* + \rho g_{t-1} + \varepsilon_t & \text{otherwise} \end{cases}$$

We assume $\varepsilon_t \sim N(0, 1.44)$, $g^* = 25$, with an upper bound $\overline{g}$ equal to 35% and a lower bound $\underline{g} = 15\%$ of average GDP and $\rho = 0.95$. $\overline{M}$ is set equal to 90% of average GDP and $\underline{M} = -\overline{M}$.

We choose $X_t^{core} = (\lambda_{t-1}, b_{N,t-1}, g_t)$ hence $X_t^{out} = (b_{N,t-2}, ..., b_{N,t-N}, \lambda_{t-2}, ..., \lambda_{t-N})$. To test if X^{core} (or subsequent linear combinations) gives a sufficiently accurate solution in Step 3 we use as our tolerance statistic:

$$dist = \frac{R^2_{aug} - R^2}{R^2}$$

where R^2 and R^2_{aug} denote the goodness of fit of the original regression based on the condensed PEA and augmented with the linear combination of residuals respectively. We use for tolerance criterion $dist \leq 0.0001$. Table 2 summarizes the number of linear combinations needed for each maturity whilst Table 3 gives more details on the role of each linear combination as it reports R^2 and $dist$.

The advantages of the condensed PEA are readily apparent. In nearly half the cases the core variables are sufficient to solve the model and at most only one linear combination of omitted variables is required. Clearly the condensed PEA can be used to solve models with large state spaces with relatively small computational cost; for example, for maturity $N = 20$ the state vector is in principle of dimension 41 but effectively a state vector of dimension of 4 is sufficient.

Whilst we have focused on a case of optimal fiscal policy and debt management this methodology clearly has much broader applications in models with large state vectors.

HERE TABLES 2 AND 3

3.4 Optimal Policy - The Impact of Maturity

3.4.1 Interest Rate Twisting

Figures 1 and 2 display the impulse response functions of key variables to an unexpected shock in g_t. The solution is computed with condensed PEA.[12] The vertical axis is in units of each of the

[12] Since debt is very persistent, to ensure we visit all possible realizations in the long run simulations of PEA we initialize the model at 9 different initial conditions, simulate it for 5000 periods for each initial condition, doing this

variables and expresses deviations from the value that would occur for the given initial condition if $g_t = g^{ss}$.[13]

Figure 1 is for the case when the government has zero debt on impact.

The only differences are on the face value of debt and interest rates. But these differences are immaterial: the face value of debt $b_{N,t}$ is obviously higher for long bonds, as long debt is discounted more heavily, so its face value needs to be higher. What is relevant is the market value of debt, which is similar. As usual in endowment models the long interest rates respond less to shocks than the short interest rate.

For all relevant purposes the differences between long and short bonds are quite minor. As usual in models of incomplete markets it is optimal to use debt as a buffer stock so that debt increases with g and it displays considerable persistence. For the case of zero debt there is no promise to reduce taxes at maturity.

HERE FIGURE 1

Figure 2 shows the same impulse response functions when we assume the government is indebted on impact as $b_{N,t-1} = 0.5\ y^*/\beta^N$ where y^* is steady state output.

HERE FIGURE 2

We see that with long bonds of maturity $N = 10$ there is a blip in taxes at the time of maturity of the outstanding bonds. This is a reflection of the promise to cut taxes with the aim to twist interest rates as discussed in Section 2.3 only that now the interest rate twisting occurs each period that there is an adverse shock if the government is in debt. The size of the promised tax cut depends on how much larger is today's shock relative to yesterday's shock $(\lambda_{t-1} - \lambda_t)$ and the level of today's debt.

3.4.2 Optimal Policy with Short Bonds

We now relate this discussion to the role of commitment with short term bonds as in Aiyagari et al (2002). Consider the case when the government is indebted when an adverse shock occurs, as in Figure 2. As we explained in Section 2.3 optimal policy is to increase current taxes but promise a tax cut in $N - 1$ periods. In the response of taxes for $N = 10$ in Figure 2 the promised tax cut is clearly distinct from the current increase in taxes. For short bonds $N = 1$ the two effects are still there but they are confounded, as they both influence taxes in the same period.

1000 times per initial condition, and compute conditional expectations discarding the first 500 observations for each simulation.

[13] Since impulse response (IR) functions are computed for initial conditions away from steady state and we have a non-linear solution we have to be clear about what we mean by these IR. We mean deviations between the values that would occur if we had a surprise shock and having the shock forever at its mean. More precisey, for the given initial condition we first compute the value of, say, consumption, that would occur if $g_t = g^{ss}$ forever. Denote this value c_t^{ss}, it is not constant because initial conditions for b_N are far from steady state mean. Then we compute the value of consumption that would occur for the realization of the shock $g_{-1} = g^{ss}$, $\varepsilon_0 = \sigma_\varepsilon$, $\varepsilon_1 = \varepsilon_2 = ... = 0$. Denote this consumption as c_t^{shock}. The impulse response for consumption at t reported in the Figures is $c_t^{shock} - c_t^{ss}$.

This is clearly seen in the response of taxes depicted in Figure 3 for maturities $N = 1, 5, 10, 20$ and different signs for initial debt. Given our previous discussion it is clear why the blip in taxes keeps moving to the left as we decrease the maturity, until the blip simply reduces the reaction of taxes on impact at $N = 1$ for the case of debt. Therefore optimal policy for short bonds if the government is in debt is to increase taxes on impact but less than would be done if considerations of interest rate twisting were absent or if debt would be zero.

Figure 3 also shows that in the case the government has assets the blip in taxes goes upwards, as the government desires to increase the value of its assets. It is clear that for short bonds the increase in taxes on impact if the govenment initially has assets is much larger than if the government is indebted.

HERE FIGURE 3

3.4.3 Second Moments

Table 4 shows second moments for the economy at steady state distribution for different maturities.[14] With the exception of debt and deficit all the moments differ only to the second or third decimal place across maturities. This may be surprising, as we have seen that tax policy does change with maturity and since we know that under incomplete markets the way government finances its expenditure can affect the real economy. However with the government only issuing one type of bond in each case tax smoothing is achieved mainly by using debt as a buffer stock rather than through fiscal insurance. The fluctuations of all variables are driven mostly by the strong low frequency fluctuations of debt, so that the interest rate twisting plays little role in these steady state second moments. We return to this issue in the discussion of Figure 6.

The main exception are the levels of debt and deficit: government in the long run holds assets, but average asset holdeings are lower for higher maturities. The mean of assets at steady state mean for 20-year bonds halves the average assets for short bonds. This is due to the different opportunities for fiscal insurance that are offered by long bonds as the following intuition shows.

As is well known, in models of optimal policy with incomplete markets there is a force pushing the government to *accumulate* long bonds in the long run. More precisely, extending the results in Aiyagari et al. (2002) Section III one can easily prove that with a linear utility of consumption $u(c) = c$ the government would purchase a very large amount of private long bonds in the long run, enough to abolish taxes. This force accounts for the negative means of debt shown in Table 4. On the other hand, as argued in Angeletos (2002), Buera and Nicolini (2004) and Nosbusch (2008), if the term premium is negatively correlated with deficits (as it is in our model) it is optimal for the government to *issue* long bonds, as this provides fiscal insurance. Hence the government is aware that accumulating a very large amount of privately issued long bonds increases the volatility of taxes. This force accounts for the lower asset accumulation with longer maturities shown in Table 4.

The lower asset holdings account for the lower primary deficits for higher N, since the value of assets is equal to the expected present value of primary deficits also under incomplete markets.

[14]These moments have been computed from very long simulations using the approximate policy function computed as described before.

Lower deficits cause taxes to be higher in steady state for higher N.

HERE TABLE 4

Varying the average maturity of debt also has an influence on the persistence of debt. Marcet and Scott (2009) (MS) argued that incomplete markets did a much better job at explaining the co-movement of debt and deficit and the persistence of debt. They measured persistence by the k-variance statistics

$$P_y^k = \frac{Var(y_t - y_{t-k})}{k\ Var(y_t - y_{t-1})}.$$

HERE FIGURE 4

The closer to 0 this measure the less persistent the variable, whereas the closer to 1 the measure the more the variable shows unit root persistence. MS showed that the observed $k-$variances for debt were even higher than 1, for example they found $P_{Debt}^{10} \simeq$2.5 in US data (see Figure 2 in MS). Values of P_{Debt}^{k} higher than one are incompatible with complete market models and optimal policy, but they easily arise under incomplete markets. But MS also report a shortcoming of incomplete markets: debt displays too much persistence under incomplete markets, as they report P_{Debt}^{10} =10 (see Figure 6 in MS).

Figure 4 shows the *small sample* mean of persistence measures for our model when the government is initially in debt.[15] A quick look shows some good news: now $P_{Debt}^{10} = 4.1$ for 20-year bonds, so the gap between the data and the model is now 1.6 (=4.1-2.5), one-fifth of the gap reported by MS (7.5=10-2.5). This vast improvement is in part due to our use of small sample moments, while MS reported $k-$variance ratios at steady state distribution. Note that even for a short maturity of $N = 2$ (and also for $N = 1$, not reported in Figure 4) we have $P_{Debt}^{10} \simeq 5$, nearly cutting the persistence in half relative to MS. With hindsight it is not surprising that the *small sample* approach of the current paper gives less persistence: it is a reflection of the standard bias in estimating auto-regressive roots in time series, where small samples always underestimate persistence. The small sample approach of this paper is clearly a better procedure, so we conclude that MS grossly oversestimated the extent to which their model fails to match persistence of debt.

Figure 4 confirms our intuition that the better fiscal insurance properties of long bonds (if the government is in debt) causes debt to be less persistent less persistent for longer bonds. We have about $P_{Debt}^{10} \simeq 4.1$ for maturity $N = 20$. Although we do not match the empirical observation of $P_{Debt}^{10} \simeq 2.5$ introducing long bonds moves this statistic significantly in the right direction.

[15] The small sample means are found by fixing initial bonds at a level of debt equal to $.5y^*$, obtain simulations of 50 periods, compute P_y^k for each realization, and average P_y^k over many realizations all starting at the same initial condition.

4 Independent Powers

The policy of interest twisting described in Sections 2 and 3 was that a cut in taxes in the distant future is announced today in order to influence current interest rates favorably. The reader may think that this optimal policy is not relevant for the "real world" because it implies too much policy commitment and too much policy coordination. Since monetary authorities are in charge of determining interest rates, fiscal authorities are in charge of taxes and debt management it is unlikely they coordinate in the way recommended by the model. Also it is unlikely that governments can credibly commit to a tax cut in the distant future. Some authors argue that full commitment is always an extreme assumption and conclude it is better to study models of discretionary policy. But assuming that the government has no possibility of committing is also an extreme assumption, as governments frequently do things only to satisfy previous announcements.

For these reasons we change the way policy is decided in this model. We relax the assumption of perfect coordination and assume the presence of a third agent, a monetary authority that fixes interest rates in every period. The fiscal authority now takes interest rates as given and implements optimal policy given these interest rates. We examine an equilibrium where the two policy powers play a dynamic Markov Nash equilibrium with respect to the strategy of the other policy power and the fiscal authority plays Stackelberg leaders with respect to the consumer. More precisely, the fiscal authority chooses taxes and debt given a sequence for interest rates, the monetary authority simply chooses interest rates that clear the market and the fiscal authority maximizes the utility of agents. This assumption sidesteps the issues of commitment, now there is no room for interest rate twisting on the part of the fiscal authority.

If the objective of the monetary authority is to choose interest rates that clear the bond market the monetary authority simply sets in equilibrium interest rates as:

$$\begin{aligned} p_{N,t} &= \frac{\beta^N E_t\left(u_{c,t+N}\right)}{u_{c,t}} \\ p_{N-1,t} &= \frac{\beta^{N-1} E_t\left(u_{c,t+N-1}\right)}{u_{c,t}}. \end{aligned} \tag{25}$$

Now the fiscal authority will not be able to manipulate interest rates, so it will loose any interest in making promises to cut future taxes. To solve this model we are looking for an interest rate policy function $\mathcal{R} : R^2 \to R^2$ such that if long interest rates at t are given by

$$(p_{N,t}^{-1}, p_{N,t-1}^{-1}) = \mathcal{R}(g_t, b_{N,t-1}) \tag{26}$$

then (25) holds and with the fiscal authority maximizing consumer utility in the knowledge of all market equilibrium conditions but taking the stochastic process for interest rates as given it chooses a bond policy such that (26) holds.

From the point of view of the fiscal authority the problem now is of the form of a standard dynamic programming program hence the vector of state variables is now $(b_{N,t-1}, g_t)$. An advantage of this model is therefore there is no reason now for longer lags to enter this state vector, as past Lagrange multipliers do not play a role. Therefore this separation of powers approach is an alternative way to reducing the state space and simplifying the solution of the model.

In this case of independent powers the Lagrangian of the Ramsey planner becomes

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \{ u(c_t) + v(x_t) + \lambda_t [S_t + p_{N,t} b_{N,t} - p_{N,t-1} b_{N,t-1}] \tag{27}$$
$$+ \nu_{1,t} \left(\overline{M} - \beta^N b_{N,t} \right) + \nu_{2,t} \left(\beta^N b_{N,t} - \underline{M} \right) \}$$

The first order condition with respect to consumption is

$$u_{c,t} - v_{x,t} + \lambda_t S_{c,t} + u_{cc,t} \lambda_t \left(p_{N,t} b_{N,t} - p_{N-1,t} b_{N,t-1} \right) = 0$$

and using the government's budget constraint gives

$$u_{c,t} - v_{x,t} + \lambda_t S_{c,t} + u_{cc,t} \lambda_t \left(g_t - \left(1 - \frac{v_{x,t}}{u_{c,t}} \right) (1 - x_t) \right) = 0 \tag{28}$$

To see the impact of Independent Powers we calibrate the model as in Section 3 and consider the case $N = 10$. Figure 5 compares the impulse responses to a one standard deviation shock to the innovation in the level of government spending when the government has debt between independent powers and the benchmark model of Section 3. As can be seen the model of independent powers does not show the blip in taxes at maturity. In this case debt management is subservient to tax smoothing and is aimed at lowering the variance of deficits.

HERE FIGURE 5

To better understand the magnitude of the interest twisting channel we can compare our independent powers model with our earlier benchmark model. We simulated the model at different time horizons $T = 40$, $T = 200$ and $T = 5000$ discarding the first 500 periods. We calculated the standard deviation of taxes for each realizations and we averaged it across simulations. We repeat the same exercise for $N = 2, 5, 10, 15, 20$. Figure 6 shows the results.

HERE FIGURE 6

In shorter sample periods the effect of twisting interest rates in connection with initial period debt is significant and provides a higher level of tax volatility in the benchmark model. Naturally as we increase the sample size the initial period effect diminishes.

The steady state second moments with independent powers are shown in Table 5. They are extremely similar to those of the benchmark model in Table 4. The additional volatility of taxes only shows up in second moments with short samples, as shown in Figure 6. We conclude that the model of independent powers may be a good model to have in the toolkit as it retains many of the interesting features of the Ramsey models, it has the same steady state moments, it avoids the technicalities arising from the very large state vector and it avoids discussion on the role to commitment at very long horizons. There are, however, issues of tax volatility showing up in small samples where the two models differ.

HERE TABLE 5

5 Hold to Redemption

With long bonds the government has a choice to make at the end of every period. It can buy back the N period bonds issued last period as assumed in Sections 2 and 3 and sell newly issued long bonds to pay for the bond repurchase. Alternatively it can leave some or all of the outstanding bonds in circulation until they mature at their specified redemption date. In models of complete markets whether or not bonds are bought back is immaterial, all prices and allocations remain unchanged. But in this paper there are two reasons why the outcome is different. The first reason is that the stream of payoffs generated by each policy is quite different from the point of view of the government: with buyback long bonds are actually one-period bonds that provide the random payoff $p_{N-1,t+1}$ next period; if the bond is held until redemption the bond pays 1 with certainty at $t+N$. As is well known, under incomplete markets not only the present value of payoffs of an asset are relevent, the timing of payoffs also matters. A second reason for the differences is that the possibilities for governments to twist interest rates are different.

In Section 2 we made the extreme assumption that the government each period buys back the whole stock of outstanding bonds issued last period. As shown in Marchesi (2004) it is normal practice for governments not to buyback debt - debt is issued and it is paid off at maturity. In this section we assume that bonds are never bought back, they are held until redemption date arrives. In the case of buyback there are only N-period bonds outstanding. In the case of holding to redemption there exist bonds at all maturities between 1 and N even though the government only issues N-period bonds.

In this section we set up the model when debt managers do not buyback debt at the end of each period, show how full commitment gives rise to a different kind of interest rate twisting, outline how to use condensed PEA to solve for optimal fiscal policy and we show the behavior of the model. Since we follow closely the analysis of Sections 2 and 3 we omit some details and focus on the differences. Although we model the implications of holding to redemption, exactly why no buyback is standard practice[16] is beyond the scope of this paper.

The economy is as before except the government budget constraint is now

$$b_{N,t-N}^{HTR} = \tau_t(1-x_t) - g_t + p_{N,t} b_{N,t}^{HTR} \tag{29}$$

so that the payment obligations of the government at t are the amount of bonds issued at $t-N$.

We include the debt limits

$$\underline{M} \leq b_{N,t}^{HTR} \sum_{i=1}^{N} \beta^i \leq \overline{M} \tag{30}$$

Again, the limit is over the value of the newly issued debt at steady state prices and steady state bond policy: if there were no shocks, government issued b_N bonds at all periods it would have b_N units of bonds of maturities 1,2,...,N outstanding so the total value of debt at steady state would be $\sum_{i=1}^{N} \beta^i b_N^{HTR}$. The budget constraint of the household's problem changes in a parallel way.

[16] Conversations with debt managers suggest some combination of transaction costs, a desire to create liquid secondary markets at most maturities or worries over refinancing risk. For simplicity we rule out a third possibility - governments choosing to only buy back a certain proportion of outstanding debt.

5.1 Optimal Policy with Maturing Debt

Substituting in equilibrium bond prices and wages net of taxes the budget constraint (29) becomes

$$\text{s.t.} \qquad b^{HTR}_{N,t-N}\ u_{c,t} = S_t + \beta^N E_t\left(u_{c,t+N}\right) b^{HTR}_{N,t} \tag{31}$$

for $t = 0, 1, ...$ The Ramsey problem is now to maximize utility (2) over choices of $\left\{c_t, b^{HTR}_{N,t}\right\}$ subject to this budget constraint and the debt limits (30) for all t. The Lagrangian becomes

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u\left(c_t\right) + v\left(x_t\right) + \lambda_t \left[S_t + \beta^N u_{c,t+N} b^{HTR}_{N,t} - b^{HTR}_{N,t-N} u_{c,t}\right] \right.$$
$$\left. + \nu_{1,t}\left(\underline{M}^{HTR} - b^{HTR}_{N,t}\right) + \nu_{2,t}\left(b^{HTR}_{N,t} - \overline{M}^{HTR}\right)\right\}$$

where λ_t is the Lagrange multiplier associated with (31), $\nu_{1,t}$ and $\nu_{2,t}$ are the ones associated with the debt limits and $\underline{M}^{HTR} \equiv \underline{M}\left(\sum_{i=1}^{N} \beta^i\right)^{-1}$, $\overline{M}^{HTR} \equiv \overline{M}\left(\sum_{i=1}^{N} \beta^i\right)^{-1}$.

The first-order conditions with respect to c_t and $b^{HTR}_{N,t}$ are

$$u_{c,t} - v_{x,t} + \lambda S_{c,t} + u_{cc,t}\left(\lambda_{t-N} - \lambda_t\right) b^{HTR}_{N,t-N} = 0 \tag{32}$$

$$E_t\left(u_{c,t+N}\lambda_{t+N}\right) = \lambda_t E_t\left(u_{c,t+N}\right) + \nu_{2,t} - \nu_{1,t} \tag{33}$$

with $\lambda_{-1} = ... = \lambda_{-M} = 0$.

In short, these FOC have two differences relative to the buyback case: in equation (32) we now have $(\lambda_{t-N} - \lambda_t)$ instead of $(\lambda_{t-N} - \lambda_{t-N+1})$ and we now have λ_{t+N} instead of λ_{t+1} in the martingale condition (33)[17].

5.2 No Uncertainty and Hold to Redemption

Let us now consider the no uncertainty case when $g_t = \overline{g}$. Proceeding in an analog way to the case of Section 2.2 we could write the implementability constraint as

$$\sum_{t=0}^{\infty} \beta^t \frac{u_{c,t}}{u_{c,0}} \widetilde{S}_t = \sum_{i=1}^{N} b^{HTR}_{N,-i}\ p_{N-i,0}\ , \text{ or} \tag{34}$$

$$\sum_{t=0}^{\infty} \beta^t S_t = \sum_{i=1}^{N} b^{HTR}_{N,-i}\ \beta^{N-i} u_{c,N-i} \tag{35}$$

for $p_{0,t} \equiv 1$. Bonds issued in periods $i = -1, -2, ..., . - N$ appropriately appear in the right side of the above constraint as what matters now is the total value of debt held in period $t = 0$.

[17] Introducing independent powers in this model would still leave a large state vector, as N lags of b_N are state variables.

Let us consider the problem of maximizing utility when (35) is the sole implementability constraint. If the government is in debt with $b_{N,-i}^{HTR} > 0$ for all $i = 1, ..., N$ it is clear that in this case interest rate twisting will involve changing interest rates in the first $N - 1$ periods hence the government will promise to cut taxes in all periods between $t = 0, ..., N-1$. The FOC for consumption indicates that the tax cut will be larger for periods $t = 0, ..., N-1$ where the maturing debt $b_{N,t-N}^{HTR}$ is larger. Therefore tax cuts now last N periods. For $t \geq N$ consumption and taxes are constant.

But assuming that (35) is the sole implementability constraint as we did in the previous paragraph is not correct for our model. This is perhaps surprising, as we usually think that without uncertainty access to one asset achieves complete markets for sufficiently high debt limits. It would be correct in a slightly different model where instead of the debt limits (30) we had a limit in the total value of debt, for example if debt limits would be

$$\underline{M}^{MV} \leq \sum_{i=1}^{N} b_{N,t-i}^{HTR} \; p_{N-i,t} \leq \overline{M}^{MV} \tag{36}$$

Take for simplicity the case $N = 2$. It is clear that the optimal allocation described in the previous paragraph can be implemented for bond issuances satisfying $b_{N,t-2}^{HTR} + \beta \frac{u_{c,t+1}}{u_{c,t}} b_{N,t-1}^{HTR} = \sum_{j=0}^{\infty} \beta^j S_{t+j}$ for all $t = 0, 1, ...$ Given initial conditions this provides a difference equation on b_N that satisfies the period-t budget constraint (31) and the value of debt limits if $\underline{M}^{MV}$ and $\overline{M}^{MV}$ were sufficiently large in absolute value.

To see that (35) is not sufficient for an equilibrium notice that the optimal allocation described above implies a constant surplus $\widetilde{S}$ for all $t \geq N$. The bonds that would satisfy the period t budget constraint satisfy $b_{N,t-2}^{NBB} + \beta b_{N,t-1}^{NBB} = \frac{\widetilde{S}}{1-\beta}$ for all $t = N, N+1, ...$ This path for bonds would satisfy the difference equation

$$b_{N,t}^{HTR} = \frac{\widetilde{S}}{(1-\beta)\,\beta} - \beta^{-1} \; b_{N,t-1}^{HTR} \;\; t = N, N+1, ... \tag{37}$$

which in general is an unstable difference equation in $b_{N,t}^{HTR}$. Normally the values of $b_{N,t}^{HTR}$ satisfying this equation will explode geometrically to plus and minus infinity, alternating sign. The sequence that is compatible with non explosive wealth of the government implies that the debt limits (30) are violated, therefore (35) is not sufficient for an equilibrium.

The intuition that one asset completes the markets for no uncertainty if the debt limits are sufficiently loose is only right if the debt limits are in terms of the value of debt (36), gross bond issuance each period would go to infinity, constant wealth is only achieved because of the alternation in signs of b_t^{HTR} each period. Of course, one modelling soltuion would be to assume the value of debt limits as in (36), but we believe limits on bonds as in (30) are the more relevant constraint. After all the bond markets are extremely concerned with gross isuance of bonds each period. Also, as discussed in Faraglia, Marcet and Scott (2010) very large values of bonds are unlikely, for example, because a large negative value of b_N implies a huge exposure of the government to private default.

Hence with long bonds and (30) we can not use (35) as the only implementability condition, we need to keep the budget constraint (31) in all periods in the analysis.

The following result shows the actual behavior of optimal policy. Essentially, we show that optimal policy induces higher tax volatility for two reasons: *i*) there are cycles of length N, *ii*) interest rate twisting is permanent, the reduction in taxes lasts N periods.

That there are $N-$period cycles follows from the following observation: the budget constraint (31) can be rolled forward as follows

$$b_{N,t-N}^{HTR} = S_t + \beta^N E_t\left(\frac{u_{c,t+N}}{u_{c,t}} b_{N,t}^{HTR}\right) = S_t + \beta^N E_t\left(\frac{u_{c,t+N}}{u_{c,t}} S_{t+N}\right) + \beta^{2N} E_t\left(\frac{u_{c,t+2N}}{u_{c,t}} b_{N,t}^{HTR}\right) = ...$$

Using debt limits we conclude

$$b_{N,t-N}^{HTR} = E_t \sum_{j=0}^{\infty} \beta^{Nj} \frac{u_{c,t+Nj}}{u_{c,t}} S_{t+Nj} \text{ for all } t = 0, 1, ... \tag{38}$$

This suggests that bonds issued at t are linked to surpluses in $t+N$, $t+2N$,... and that a shock today will echo each N periods. Indeed this is formalized in the following

Result 2. *Assume $b_{N,-i}^{HTR} > 0$ for all $i = 1, ..., N$* and no uncertainty $g_t = \overline{g}$. *Optimal policy for the model in this section shows cycles of order N in taxes and in bonds. More precisely*

$$\tau_i = \tau_{tN+i} \qquad i = N, ..., 2N-1 \text{ for all } t = 1, 2, ...$$

and

$$b_{N,i}^{HTR} = b_{N,tN+i}^{HTR} \quad i = 0, ..., N-1, \text{ for all } t = 1, 2, ...$$

Assume further the standard utility function where higher λ (in a complete markets case) would imply lower taxes, as for example happens with the utility (18), then

$$\tau_{i+N} > \tau_i \quad i = 0, ..., N-1$$

Furthermore, if $b_{2,-2}^{HTR} > b_{2,-1}^{HTR}$ then $\tau_0 < \tau_1$

Proof

Consider the case $N = 2$. It is clear from the martingale condition (33) that

$$\begin{aligned} \lambda_t &= \lambda_0 \text{ for all } t > 0, \ t \text{ even} \\ \lambda_t &= \lambda_1 \text{ for all } t > 1, \ t \text{ odd} \end{aligned}$$

Therefore

$$\begin{aligned} u_{c,t} - v_{x,t} + \lambda_0 S_{c,t} &= 0 \text{ for all } t \text{ even, } t \geq 2 \\ u_{c,t} - v_{x,t} + \lambda_1 S_{c,t} &= 0 \text{ for all } t \text{ odd, } t \geq 3 \end{aligned} \tag{39}$$

notice the only difference between even and odd is in the lagrange multiplier λ. This proves

$$\begin{aligned} c_t &= c_2, \ \tau_t = \tau_2 \text{ for all } t > 2, \ t \text{ even} \\ c_t &= c_3, \ \tau_t = \tau_3 \text{ for all } t > 3, \ t \text{ odd} \end{aligned} \tag{40}$$

Equation (38) combined with (40) implies

$$\begin{aligned}
b_t^{HTR} &= b_0^{HTR} = \frac{S_2}{1-\beta^2} \text{ for all } t \geq 0,\ t \text{ even} \\
b_t^{HTR} &= b_1^{HTR} = \frac{S_3}{1-\beta^2} \text{ for all } t \geq 1,\ t \text{ odd}
\end{aligned}$$

The only statement left to prove are the tax cuts in periods $t = 0, 1$. For periods $t = 0, 1$ we have

$$\begin{aligned}
u_{c,0} - v_{x,0} + \lambda_0 S_{c,0} - u_{cc,0}\ \lambda_0\ b_{2,-2}^{HTR} &= 0 \\
u_{c,1} - v_{x,1} + \lambda_1 S_{c,1} - u_{cc,1}\ \lambda_1\ b_{2,-1}^{HTR} &= 0
\end{aligned}$$

Notice that the difference with (39) for $t > 1$ is the presence of the terms $u_{cc,0}\ \lambda_0\ b_{2,-2}^{HTR}$ and $u_{cc,1}$ $\lambda_1\ b_{2,-1}^{HTR}$. These are clearly negative, implying that for the considered utility functions we have

$$\begin{aligned}
\tau_2 &> \tau_0 \\
\tau_3 &> \tau_1
\end{aligned}$$

The statement in the last line follows immediately from the last FOC written. ■

These results could be easily extended to the case of uncertainty only in period $t = 1$ as in Section 2.3.1, to show that if an adverse shock to g occurs taxes are lowered for the next $N - 1$ periods and there is a cycle of order N.

5.3 Numerical solutions

To write the model recursively we observe that the Lagrangean can be rewritten as

$$\begin{aligned}
L = E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ & u(c_t) + v(x_t) + \lambda_t S_t + u_{c,t} (\lambda_{t-N} - \lambda_t)\, b_{N,t-N}^{HTR} \\
& + \nu_{1,t} \left(\underline{M}^{HTR} - b_{N,t}^{HTR} \right) + \nu_{2,t} \left(b_{N,t}^{HTR} - \overline{M}^{HTR} \right) \Big\}
\end{aligned} \tag{41}$$

for $\lambda_{-1} = ... = \lambda_{-N} = 0$. In a recursive formulation we would have the $2N + 1$ states $\left[\lambda_{t-1}, ..., \lambda_{t-N}, b_{N,t-1}^{HTR}, ..., b_{N,t-N}^{HTR}, g_t \right]$ just as before. We use condensed PEA again. The FOC show that this problem is easier to solve than the one in Section 3 as there are only two expectations to approximate, $E_t (u_{c,t+N} \lambda_{t+N})$, and $E_t (u_{c,t+N})$. We choose the core $X_t^{core} = (\lambda_{t-N}, b_{N,t-N}^{HTR}, g_t)$. We keep the same tolerance level as in the model with buy back. Table 6 summarizes the number of linear combinations we needed to approximate our expectations. Relative to Section 3.3 the state space required effectively is larger - in some cases we need to add two linear combinations of residuals. Effectively this just means a total of five state variables is enough. The condensed PEA still dramatically reduces the state space and it makes computation of a non-linear solution feasible.

Figure 7 shows the impulse response functions for a 10 period bond under hold to redemption and buyback with the same calibration as in the previous sections. We see from the impulse response functions for tax rates that varying the maturity of the bond does affect optimal policy, even for initial zero debt.

INSERT FIGURE 7

The impulse response for the buyback case (which is the same as in Figure 1) the government does not promise a cut in taxes when initial debt is zero. Only when the government is in debt $b_{N,-1} > 0$ (or has assets), as in Figure 2 (or the second panel of Figure 3), we observed the promise to cut (increase) taxes in $N-1$ periods. Figure 7 however shows that even in the case of zero initial debt taxes show fluctuations. Taxes increase on impact, the response is decreasing for $N-1$ periods, then it jumps at the time of maturity to start going back down after that and so on. The positive but decreasing response for the first $N-1$ periods is standard in optimal taxation models with serially correlated shocks, it would also occur under complete markets: the higher g_t on impact indicates that g_t will also be higher in the next periods, and this generates higher taxes for the next few periods for the utility function considered. The jump in the response function at lag N is a reflection of the fact that there are cycles of order N, as suggested by Result 2 and as can be seen directly from the martingale condition (33), and as suggested by (38). Strictly speaking λ is not a risk-adjusted martingale but one can say that it is a risk-adjusted martingale of cycle N[18]. The initally high but decreasing response echoes N periods later.

The reason for these cycles lies in the link between today's issued bonds and the surpluses in $N, 2N, 3N, ...$ periods from now highlighted by equation (38). If today we have a bad shock and we issue $N-$period bonds, when these bonds mature N periods from now there will be a need for higher taxes and a higher deficit, so $b_{N,t+N}$ will increase hence there will be a need for higher taxes and higher deficits in $2N$ periods and so on. Therefore it is reasonable that there is a cycle of period N and that optimal policy has the shape displayed in Figure 7. The optimal response to an unexpected shock is to promise future taxes that in part accomodate the additional debt servicing in the periods when today's debt will have to be repaid.

Of course, the government would prefer to smooth taxes and to put some of the burden of a high g_t in a higher tax at $t+1$ or $t+2$. But it does not have the financial instrument to do so in this model, instead it can only smooth taxes across periods $t, t+N, t+2N, ...$ causing the $N-$period cycle to arise

Result 2 suggests that taxes in the first $N-1$ periods should be lower if the government is in debt. This suggests that optimal policy will be to lower taxes during the first cycle of N periods relative to later cycles. An additional role of commitment is indeed to promise a cut in taxes during the first cycle relative to the cycles later down the line. This is why in Figure 9, which looks at the case of initial debt, the main difference with Figure 8 is that the second peak in taxes is lower than the first peak, while the opposite is true in Figure 8.

HERE TABLE 7

[18] Formally, letting $\xi_t^i = \lambda_{i+tN}$ for $i = 0, 1, ..., N-1$, each ξ_t^i is a risk-adjusted martingale.

Table 7 shows summary statistics for the model with no buyback and bonds of varying maturities. The results are exceptionally similar to the case of buyback. Because debt is held to maturity each period the government now issues fewer bonds per period. As in the no buyback case the short sample second moments do show more volatility of tax rates, as shown in Figure 8.

INSERT FIGURE 9

6 Conclusions

This paper has had two interrelated aims. The first has been to study optimal fiscal policy when governments issue bonds of long maturity. The second has been to propose a general method for solving models with a large state space - the condensed PEA.

A number of additional considerations arise when governments issue long term bonds. If the government inherits debt it has an incentive to twist interest rates to minimize costs of funding debt. This is achieved by violating tax smoothing and promising a tax cut in $N-1$ periods, when the existing bonds mature. Interestingly, a typical debt management concern, namely lowering the cost of debt shapes the path of fiscal policy. This suggests that it is important to consider debt management and fiscal policy jointly. This commitment to cut taxes leads to time inconsistency.

The model with long bonds helps to clarify the role of commitment in models of optimal fiscal policy and incomplete markets, as the promised tax cut takes place in a different period than the change in taxes on impact. In the case of short bonds the change in taxes needed to adjust to a shock and the promise to cut taxes at time of maturity are conjoined, optimal taxes increase on impact much less if the government is in debt and much more if the government has assets.

This commitment to cut taxes leads to a potentially very large state space of dimension $2N+1$. Using the condensed PEA enables us to solve this model accurately with a much reduced state space allowing for computation of non-linear numerical solutions.

We have also proposed an alternative model of government policy, one where a central bank determines interest rates, and a fiscal authority separately decides on debt and taxes. This model of independent powers is of interest per se, as policy authorities may not be able to coordinate as much as is required to implement the full commitment solution. Also, it does not display policies where promises that will be implemented very far in the future matter for today's solution. As such it serves to highlight the role of commitment and to look at a solution when state space is not enormous.

We started with the case usually considered in the literature where government buys back the existing stock of debt each period. To get closer to actual practice we study the case where government bonds are left in circulation until maturity. This model gives rise to even more tax volatility due to debt management concerns: promises to cut taxes for interest twisting purposes are now permanent and policy creates $N-$period cycles, giving rise to even more tax volatility.

There is little quantitative difference in fiscal policy or economic allocations at steady state second moments as the maturity of debt is varied, justifying the observation in Table 1 that similar countries may have very different average maturity of debt. The main difference is in the steady state of debt: longer maturities imply lower asset accumulation because long bonds provide a volatile deficit if the government holds assets. However, for second moments computed with short run moments we do find more tax volatility with long bonds.

A number of further issues remain. We have throughout this paper assumed the government can issue only one bond and have varied its maturity. In order to fully understand debt management we need to consider the case when the government can issue several bonds of different maturity and choose the optimal portfoliio. Another important issue is to consider why governments do not buyback debt – presumably because of concerns over transaction costs. We have abstracted from crucial elements of actual debt management practice such as refinancing risk, rollover risk, transaction costs, default, etc., We hope the methodologies of this paper will enable us to provide a detailed study of optimal debt management and to introduce some of these features in the analysis.

References

[1] Aiyagari, R.,Marcet, A., Sargent, T.J. and Seppälä, J. (2002) "Optimal Taxation without State-Contingent Debt", *Journal of Political Economy*, 110, 1220-1254

[2] Angeletos, G-M (2002) "Fiscal policy with non-contingent debt and optimal maturity structure", *Quarterly Journal of Economics, 27, 1105-1131*

[3] Arellano, C and Ramanarayanana (2008) "Default and the Maturity Structure in Sovereign Bonds", University of Minnesota mimeo

[4] Barro, R.J. (1979) "On the determination of the public debt", *Journal of Political Economy*, 87, 940-71.

[5] Barro, R.J (2003) "Optimal Management of Indexed and Nominal Debt" *Annals of Economics and Finance,4, 1-15*

[6] Bohn, H (1990) "Tax smoothing with financial instruments" , *American Economic Review*, Vol. 80(5), pp. 1217–30.

[7] Broner, F, Lorenzoni, G and S.Schmulker (2007) "Why do emerging markets borrow short term?" MIT working paper

[8] Buera F. and J.P. Nicolini (2004) "Optimal Maturity of Government Debt with Incomplete Markets", *Journal of Monetary Economics, 51, 531-554* .

[9] Debt Management Office (2003) "DMO Annual Review 2003/4" HM Treasury

[10] den Haan, W. and Marcet, A. (1990) "Solving the stochastic growth model by parameterizing expectations", *Journal of Business and Economic Statistics*, 8, 31-34.

[11] Faraglia, E, Marcet, A and A.Scott (2008) "Fiscal Insurance and Debt Management in OECD Economies", The Economic Journal, (118) 363-386

[12] Faraglia, E., Marcet, A and A. Scott (2010); "In Search of a Theory of Debt Management" *Journal of Monetary Economics,* 57, 821–836

[13] Lucas, R.E. and Stokey, N.L. (1983) "Optimal Fiscal and Monetary Policy in an Economy without Capital", *Journal of Monetary Economics*, 12, 55-93.

[14] Lustig, H., Sleet, C., Yeltekin, S., (2009) "Fiscal hedging with nominal assets" *Journal of Monetary Economics* 55, 710–727

[15] Marcet, A and Marimon, R (2011) "Recursive Contracts", working paper, Institut d'Anàlisi Econòmica CSIC, Universitat Autònoma de Barcelona.

[16] Marcet, A and Scott (2009) "Debt and Deficit Fluctuations and the Structure of Bond Markets" *Journal of Economic Theory* 144 (2009) 473–501

[17] Marchesi (2004) "Buybacks of domestic debt in public debt management", University of Siena mimeo

[18] Nosbusch, Y (2008) ”Interest Costs and the Optimal Maturity Structure of Government Debt”, *Economic Journal,* 118, 477–498

[19] Woodford, M (2001) "Fiscal Requirements of Price Stability" *Journal of Money Credit and Banking* 33 669-728

Table 1 - Average Maturity Government Debt 2010

Country	Average Maturity (Years)
UK	13.7
Denmark	7.9
Greece	7.7
Italy	7.2
Austria	7
France	6.9
Ireland	6.8
Spain	6.7
Switzerland	6.7
Portugal	6.5
Czech Republic	6.4
Sweden	6.4
Germany	5.8
Belgium	5.6
Japan	5.4
Netherlands	5.4
Canada	5.2
Poland	5.2
Australia	5
Norway	4.9
US	4.8
Finland	4.3
Hungary	3.3

Source : OECD, The Economist

Table 2: Model with buyback -
linear combinations introduced with Condensed PEA

N	$2N+1$	# of linear comb.		
		$\mathbf{\Phi}_{\lambda}$	$\mathbf{\Phi}_{uc_N}$	$\mathbf{\Phi}_{uc_{N-1}}$
1	3	-	-	-
2	5	0	1	0
5	11	0	1	0
10	21	0	1	0
15	31	1	1	0
20	41	1	1	1

Note: recall that N denotes maturity and $2N+1$ is the dimension of the state vector. In all cases X^{core} has three variables. "# of linear comb" refers to how many linear combinations of X^{out} had to be added to satisfy the accuracy criterion. We denote each expectation to be approximated by $\Phi_{\lambda} = E_t\left(u_{c,t+N}\lambda_{t+1}\right)$, $\Phi_{uc_N} = E_t\left(u_{c,t+N}\right)$ and $\Phi_{uc_{N-1}} = E_t\left(u_{c,t+N-1}\right)$

Table 3: Model with buyback -
accuracy measures in Condensed PEA

		adding 1 linear comb			adding 2d linear comb		
N		$\mathbf{\Phi}_{\lambda}$	$\mathbf{\Phi}_{uc_N}$	$\mathbf{\Phi}_{uc_{N-1}}$	$\mathbf{\Phi}_{\lambda}$	$\mathbf{\Phi}_{uc_N}$	$\mathbf{\Phi}_{uc_{N-1}}$
2							
	# lin comb in	0	0	0	0	1	0
	$\mathbf{R}^2_{aug}$	0.9208	0.7533	0.8669	0.9209	0.7535	0.8669
	dist	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000
5							
	# lin comb in	0	0	0	0	1	0
	$\mathbf{R}^2_{aug}$	0.9069	0.5022	0.5751	0.9070	0.5026	0.5754
	dist	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000
10							
	# lin comb in	0	0	0	0	1	0
	$\mathbf{R}^2_{aug}$	0.8911	0.2630	0.2991	0.8909	0.2632	0.2993
	dist	0.0000	0.0002	0.0000	0.0000	0.0000	0.0000
15							
	# lin comb in	0	0	0	1	1	0
	$\mathbf{R}^2_{aug}$	0.8814	0.1422	0.1609	0.8831	0.1446	0.1635
	dist	0.0001	0.0002	0.0000	0.0000	0.0000	0.0000
20							
	# lin comb in	0	0	0	1	1	1
	$\mathbf{R}^2_{aug}$	0.8751	0.0788	0.0886	0.8771	0.0807	0.0907
	dist	0.0002	0.0003	0.0002	0.0000	0.0000	0.0000

Note: see Note of previous Table. R^2_{aug} and $dist$ are defined in section 3.3.

Table 4: Second Moments, Steady State
Model: **Ramsey with buyback**

	N	c	y	τ	**deficit**	R_N	**MV** $=p_N b_N$	$\boldsymbol{\lambda}$
mean								
	1	52.60	70.11	0.243	0.42	2.02	-24.68	0.057
	5	52.58	70.08	0.245	0.32	2.02	-19.21	0.058
	10	52.56	70.06	0.246	0.25	2.03	-16.28	0.058
	20	52.54	70.05	0.247	0.17	2.03	-12.46	0.059
std								
	1	3.49	0.35	0.044	1.46	0.5	27.26	0.013
	5	3.48	0.37	0.043	1.57	0.4	30.96	0.013
	10	3.48	0.38	0.044	1.59	0.3	31.97	0.013
	20	3.48	0.39	0.044	1.66	0.2	32.84	0.014

Note: to provide a more interpretable quantity we report annualized interest rates instead of bond prices, namely $R_N = \left((p_N)^{-\frac{1}{N}} - 1\right) 100$.

Table 5: Second Moments, Steady State
Model: **Independent Powers**

	N	c	y	τ	**deficit**	R_N	**MV** $=p_N b_N$	$\boldsymbol{\lambda}$
mean								
	1	52.60	70.10	0.244	0.41	2.02	-23.54	0.057
	5	52.58	70.08	0.245	0.32	2.02	-19.49	0.058
	10	52.56	70.07	0.246	0.26	2.03	-16.40	0.058
	20	52.54	70.05	0.247	0.17	2.03	-12.31	0.059
std								
	1	3.49	0.34	0.044	1.43	0.5	27.88	0.013
	5	3.48	0.36	0.044	1.51	0.4	31.11	0.013
	10	3.48	0.37	0.044	1.54	0.3	32.20	0.013
	20	3.49	0.37	0.044	1.56	0.2	33.20	0.014

Table 6: Model without buyback

N	$2N+1$	# lin. comb. in Φ_λ	Φ_{uc_N}
1	3	-	-
2	5	0	0
5	11	0	0
10	21	2	2
15	31	2	2
20	41	2	2

Note: same as in Table 2 except we denote expectations to be approximated by $\Phi_\lambda = E_t\left(u_{c,t+N}\lambda_{t+N}\right)$, $\Phi_{uc_N} = E_t\left(u_{c,t+N}\right)$.

Table 7: No Buy Back Model with Different Maturities

	maturity	c	y	τ	deficit	$\mathbf{R}_N$	MV	$\boldsymbol{\lambda}$
average								
	1	52.60	70.11	0.243	0.43	2.02	-24.69	0.057
	5	52.57	70.07	0.246	0.28	2.02	-17.43	0.058
	10	52.55	70.05	0.247	0.22	2.03	-14.53	0.058
	20	52.54	70.05	0.247	0.19	2.03	-12.77	0.059
std								
	1	3.49	0.35	0.044	1.46	0.5	27.26	0.013
	5	3.47	0.40	0.044	1.67	0.4	32.26	0.014
	10	3.48	0.41	0.044	1.72	0.3	33.98	0.014
	20	3.50	0.41	0.046	1.71	0.2	33.81	0.015

Figure 1: Responses to a shock in g_t, Benchmark model
Maturities 1 and 10: $b_{N,-1} = 0$

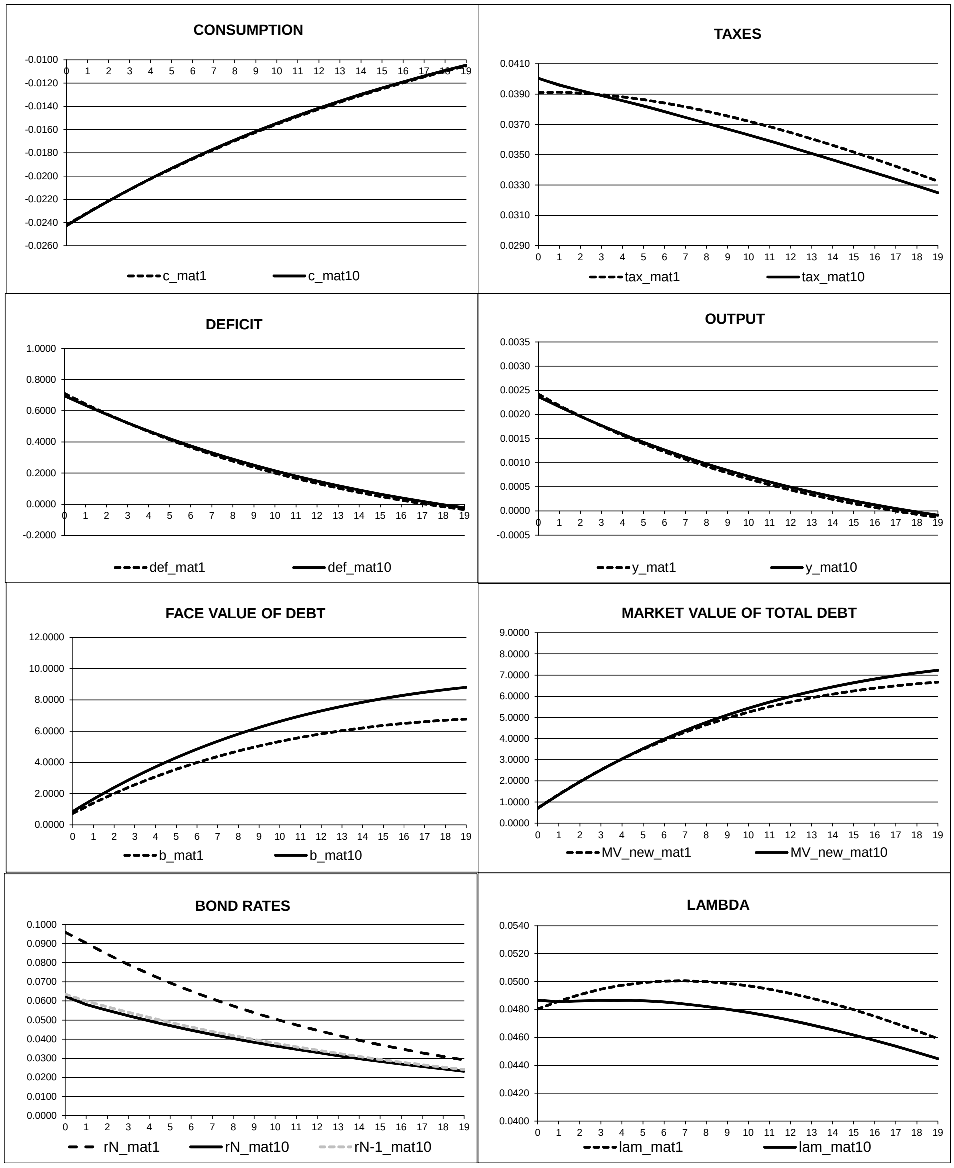


Figure 2: Responses to a positive shock in g_t, Benchmark model Maturities 1 and 10: $b_{N,-1} = \frac{0.5y^*}{\beta^N}$

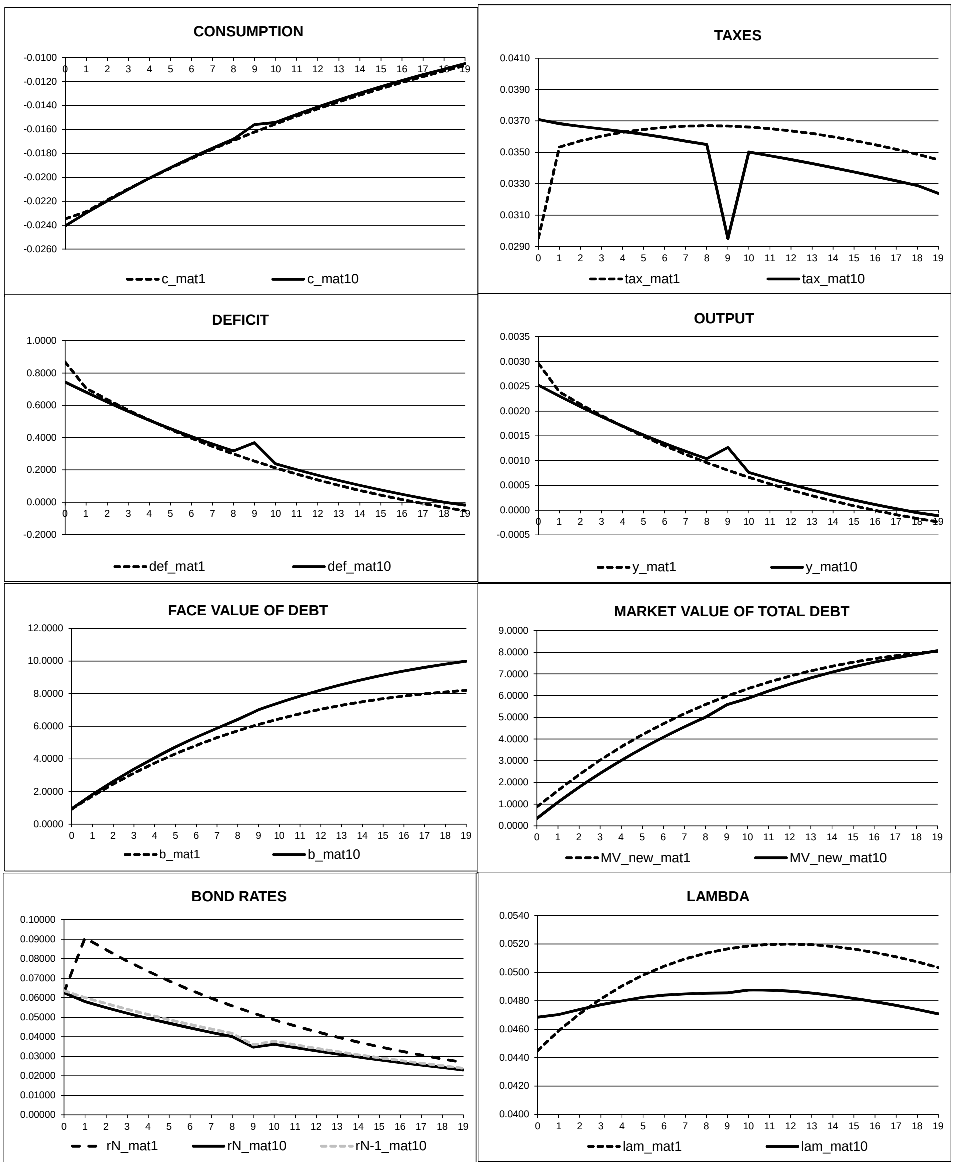
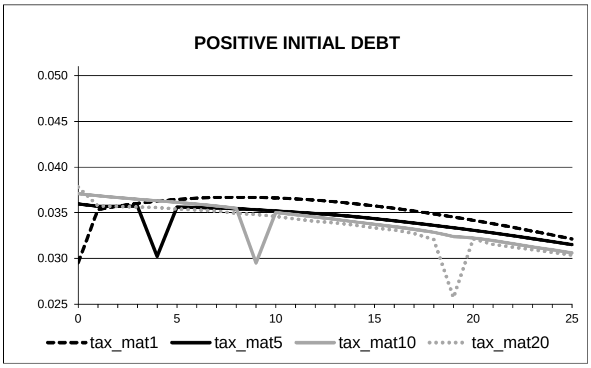


Figure 3: Responses to a positive shock in g_t, Benchmark model
Maturities 1, 5, 10 and 20:
Taxes: $b_{N,-1} = \frac{0.5y^*}{\beta^N}$



Taxes: $b_{N,-1} = -\frac{0.5y^*}{\beta^N}$

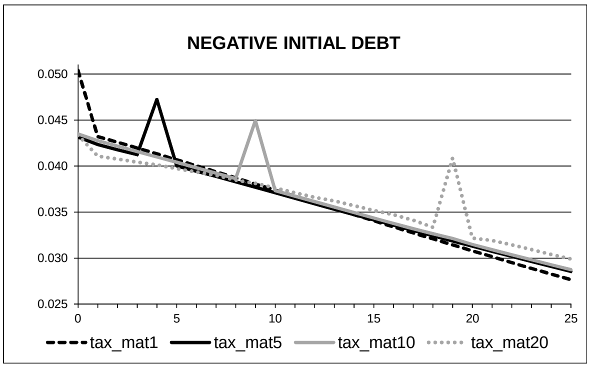


Figure 4: k-Variances, Benchmark model
Maturities 2, 5, 10 and 20

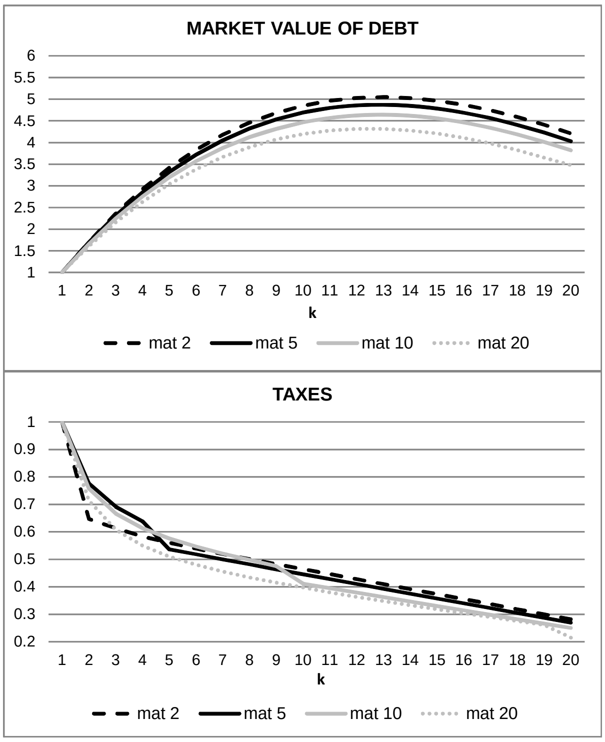


Figure 5: Responses to a positive shock in g_t, Benchmark and Independent Power Model
Maturity 10: $b_{N,-1} = \frac{0.5y^*}{\beta^N}$

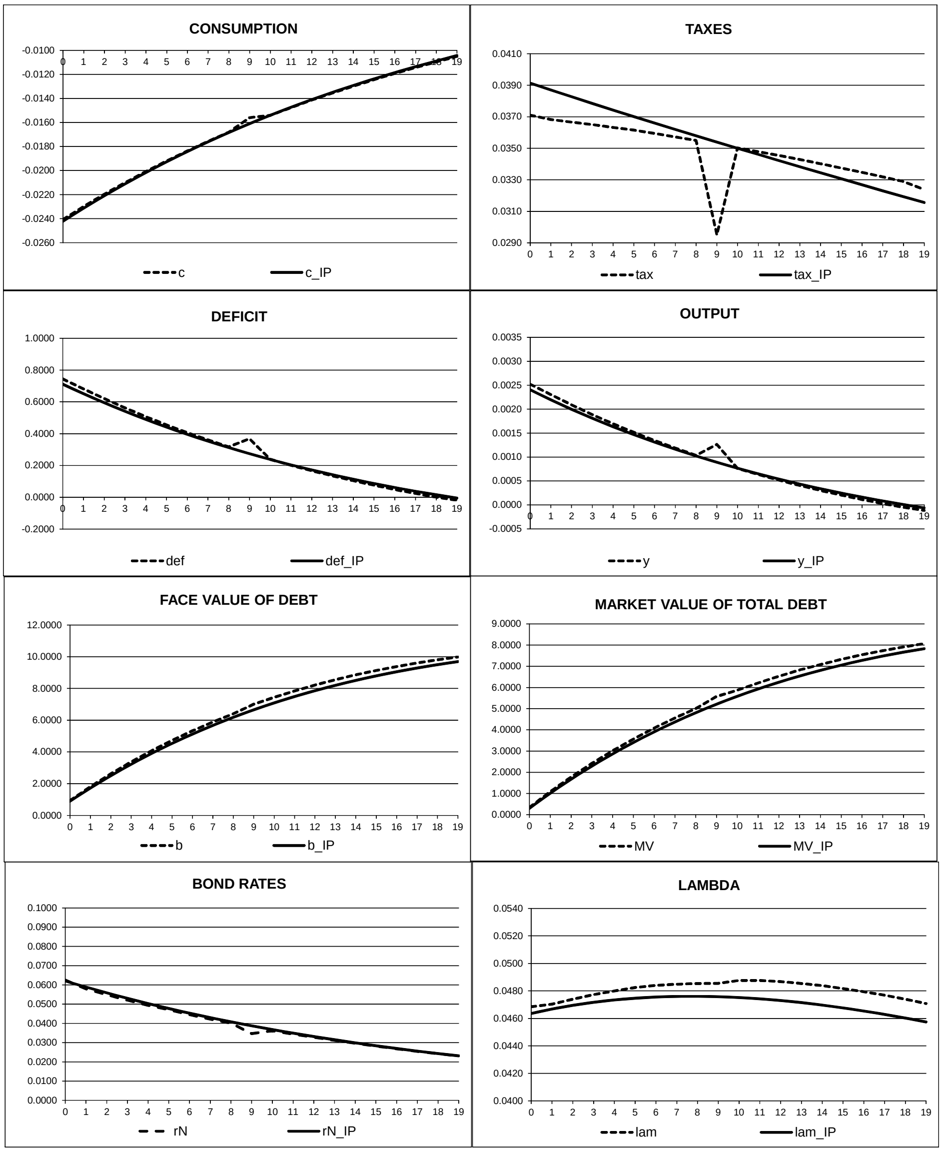


Figure 6: Tax Volatility at Different Horizons
Benchmark and Independent Powers Model

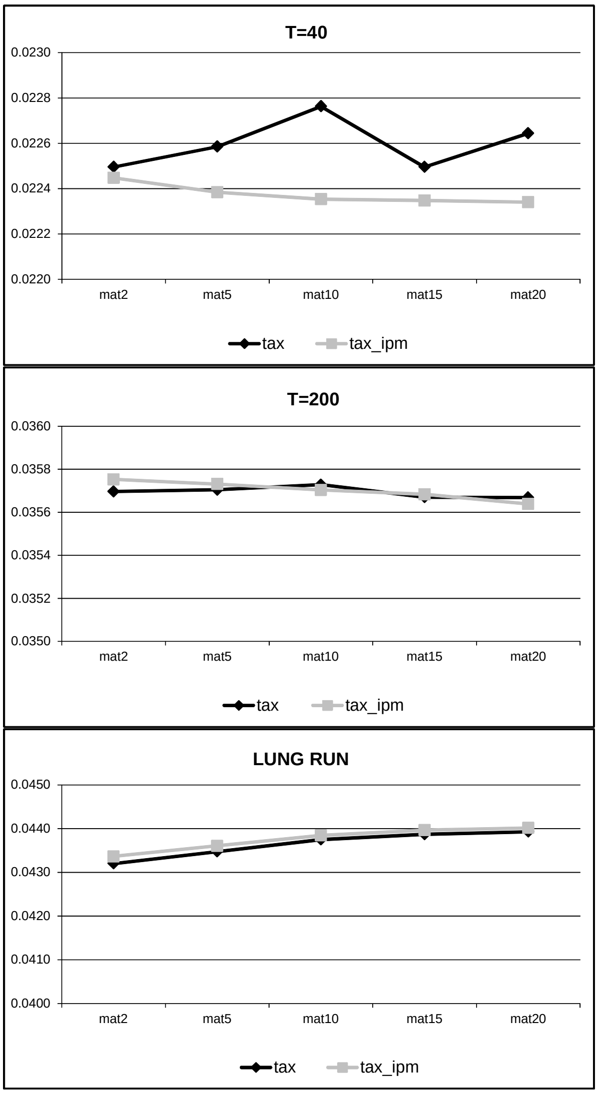


Figure 7: Responses to a positive shock in g_t, Benchmark and Holding to Redemption Model
Maturity 10: $b^{HTR}_{N,-1} = ... = b^{HTR}_{N,-1} = 0$

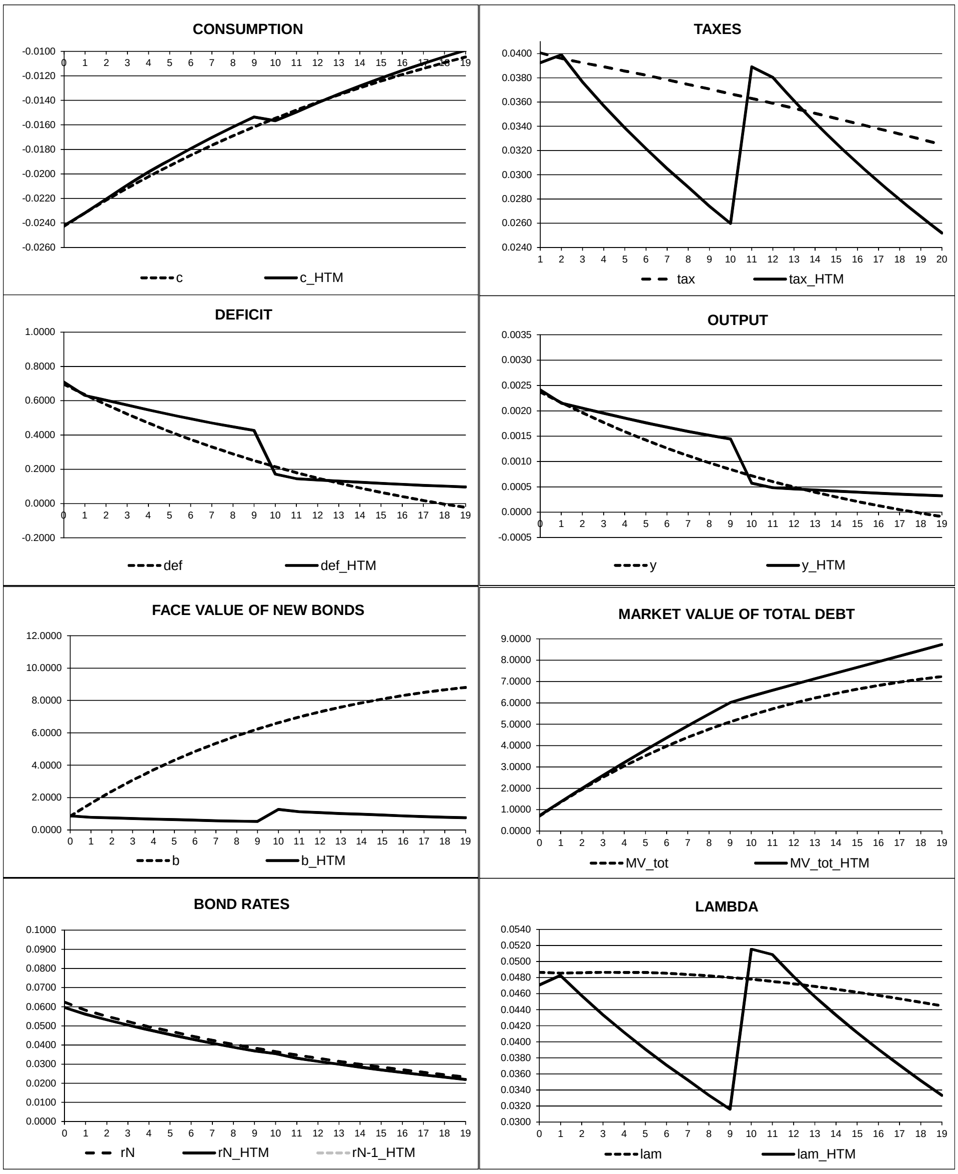


Figure 8: Tax Volatility at Different Horizons
Benchmark and Holding to Redemption Model

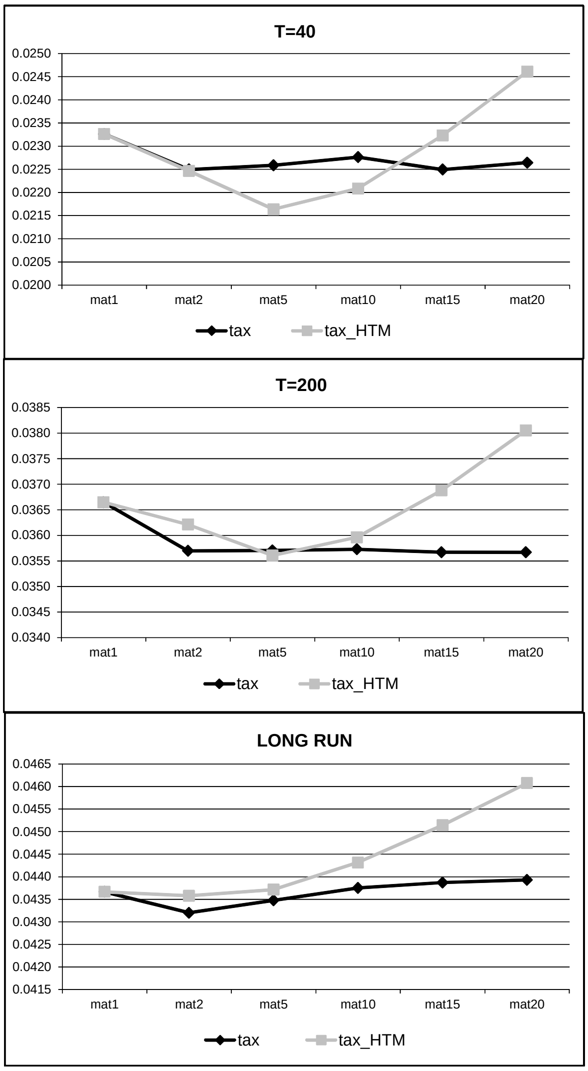


Figure 9: Responses to a positive shock in g_t, Benchmark and Holding to Redemption Model

Maturity 10: $b_{N,-1}^{HTR} = ... = b_{N,-1}^{HTR} = \frac{0.5 y_{ss}}{\sum_{i=0}^{N-1} \beta^i}$