

# A Search Story About the Peacock's Tail

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DO NOT READ AND CIRCULATE!

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## Abstract

This paper analyzes a biological model, where (i) males can have an observable and disadvantageous genetic attribute, (ii) females differ in some stochastic endowments, and (iii) screening partners is costly. The main result of the paper is that each male has the disadvantageous attribute in the unique evolutionary equilibrium, whenever the endowment of the females and the attribute of the males are substitutes in the reproduction function.

## 1 Introduction

This note proposes a new theory explaining how disadvantageous physical male attributes can be the result of an evolutionary process. The theory is based on two assumptions: (1) screening partners is costly for the females, and (2) females have stochastic quality which substitutes for the disadvantageous attribute of the males.

We show that there are two equilibria. In one equilibrium, each male has the disadvantageous attribute and each female is willing to mate with these males. In the other equilibrium, none of the males have the disadvantageous attribute and each female mates with them. The main result is that the first equilibrium is stable but the second one is unstable.

The intuition behind this result can be explained as follows. Suppose that each male has the disadvantageous attribute and a small fraction of mutant males without the attribute appears. Since the quality of the females substitutes for the attribute, low-quality females have more incentive to screen the males. This implies that the mutant male will likely to mate with a low-quality female and, as a result, have fewer number of offspring than the rest of the males. In other words, intense screening by low-quality females kills off these mutations and stabilizes the equilibrium. Similarly, if none of the males have the attribute, a mutant male with the attribute is likely to end up mating with high-quality females. Again, this is because high-quality females have less incentive to screen and more likely to end up with a low quality male. This implies that the mutant male is likely to reproduce more than the rest of the males. That is, the low incentive to screen by the high-quality females destabilize the efficient equilibrium.

## 2 Model

Time is discrete and there is a continuum of individuals. Each individual lives for one period. An individual is either male or female. Males have one of two observable attributes:  $a$  and  $d$ . (Attribute  $a$  is advantageous and attribute  $d$  is disadvantageous.) These attributes are genetic and passed on to male offspring. That is, if a male has attribute  $c (c \in \{a, d\})$  then all of his male offspring will have attribute  $c$ . Females are identical at birth but they receive a random endowment  $E$ . We normalize the distribution of  $E$  to be uniform on  $[0, 1]$ . This endowment is iid across individuals and across generations.

*Reproduction.*— Individuals are matched according to a process specified below, reproduce and die. If a male with attribute  $c (c \in \{a, d\})$  mates with a female with endowment  $E$  they reproduce  $2q(c, E)$  number of offspring half of which is male and the other half is female. The function  $q(c, E) (c \in \{a, d\})$  is positive, strictly increasing and concave in  $E$ , and continuously differentiable in  $E$ . In addition we make the following assumptions:

- A1.  $q(a, E) > q(d, E)$  for all  $E \in [0, 1]$ .
- A2.  $q(d, E) / q(a, E)$  is increasing in  $E$ .

Note that Assumption 1 means that  $a$  is indeed an advantageous attribute. Assumptions 2 means that the attribute of the males and the endowment of the females are substitutes. The following two assumptions require that the disadvantageous attribute is not too disadvantageous.

- A3.  $q(a, E) < E(q(d, E') : E' \geq E)$  for all  $E \in [0, 1]$ .
- A4.  $1 / [\partial \lg q(a, E) / \partial E] - 1 / \partial \lg q(d, E) / \partial E \leq 1/2$  for all  $E \in [0, 1]$ .

Assumption 3 means that an  $a$ -male who mates with a female with  $E$  has less offspring than a  $d$ -male who randomly mates a female among those who have endowments larger than  $E$ . This means that the disadvantage of a male can be fully offset by the high endowment of a female. An immediate consequence of A1 and A3 is  $q(a, 1) = q(d, 1)$ .

Assumption 4 requires more explanation. Note that  $\partial \lg q(c, E) / \partial E (c \in \{a, d\})$  measures how fast the function  $q(c, \cdot)$  grows. Since  $q(a, 1) = q(d, 1)$  and  $q(a, 0) > q(d, 0)$  the function  $q(d, \cdot)$  has to grow faster than  $q(a, \cdot)$ . Therefore,  $1 / [\partial \lg q(a, E) / \partial E] - 1 / [\partial \lg q(d, E) / \partial E] > 0$  for some  $E$ s.<sup>1</sup> Of course, the closer the two functions are to each others the smaller is this difference. Assumption

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<sup>1</sup>To see this, note that

$$\begin{aligned} q(d, 1) &= q(d, 0) e^{\int_0^1 \frac{q_2(d, E)}{q(d, E)} dE} \\ &= q(a, 0) e^{\int_0^1 \frac{q_2(a, E)}{q(a, E)} dE} = q(a, 1). \end{aligned}$$

Since  $q(d, 0) < q(a, 0)$  it follows that

$$\int_0^1 \frac{q_2(d, E)}{q(d, E)} dE > \int_0^1 \frac{q_2(a, E)}{q(a, E)} dE,$$

and hence,  $q_2(d, E) / q(d, E) > q_2(a, E) / q(a, E)$  for some  $E \in [0, 1]$ .

4 requires that the difference between the growth rates of the two functions is less than a half, that is, the difference between the two functions is not too large.

*Matching.*— There are two matching markets, one for  $a$  and one for  $d$ . If there are  $m$  males and  $f$  females in a market then  $\min\{f, m\}$  are matched of each gender and the rest is unmatched. (Suppose  $f > m$ . Then the probability of being matched is  $m/f$  for a female and one for a male.  $\lambda(n, c)$  is the probability of matching if the gender is  $n$  and the marriage market is for  $c$ . The only decision in this model is which market to enter. (Males make no decisions.)

*Equilibrium.*— The strategy of a female determines which market she enters as a function of her endowment. We assume that the strategy of a female is genetically determined, that is, the strategy of a female is identical to her mother's strategy.

In each period, the environment described above can be described by the fraction of males who have attribute  $d$  and the population strategy of the females. Let  $\mu_t$  denote the fraction of males who have attribute  $d$  in the male population, and  $\mathcal{S}_t$  the distribution of strategies of the females at time  $t$ . We call the pair  $(\mu, \mathcal{S})$  a steady state if the both  $\mu$  and  $\mathcal{S}$  is constant over time if the initial condition is  $(\mu, \mathcal{S})$ . Formally,

**Definition 1** *The pair  $(\mu, \mathcal{S})$  is a steady state if  $(\mu_t, \mathcal{S}_t) = (\mu, \mathcal{S})$  for all  $t \geq 1$  whenever  $(\mu_0, \mathcal{S}_0) = (\mu, \mathcal{S})$ .*

We call a strategy a best-response  $(\mu, \mathcal{S})$  if it maximizes the number of offspring given  $(\mu, \mathcal{S})$ . We say that the strategy of a female,  $s$ , is a best-response to  $(\mu, \mathcal{S})$  if  $s(E)$  maximizes the number of offspring for each  $E$ . Notice that A2 implies that if a female with endowment  $E$  finds it optimal to enter into the market for  $d$  then she would also find it optimal to enter into this market with a higher endowment. Hence, the best-response against each  $(\mu, \mathcal{S})$  is unique and defined by a cutoff. Therefore, in what follows, we shall restrict attention to cutoff strategies. Let  $E^*$  denote the cutoff of the females above which they enter into the market for  $d$ . With a slight abuse of notation, for each  $E \in [0, 1]$ , let  $E$  also denote the population strategy where each female goes to market  $d$  if and only if her endowment is smaller than  $E$ .

**Definition 2** *We call an  $(\mu, E^*)$  a steady-state equilibrium if (1)  $E^*$  is a best response to  $(\mu, E^*)$  and (2)  $\mu$  is constant over time.*

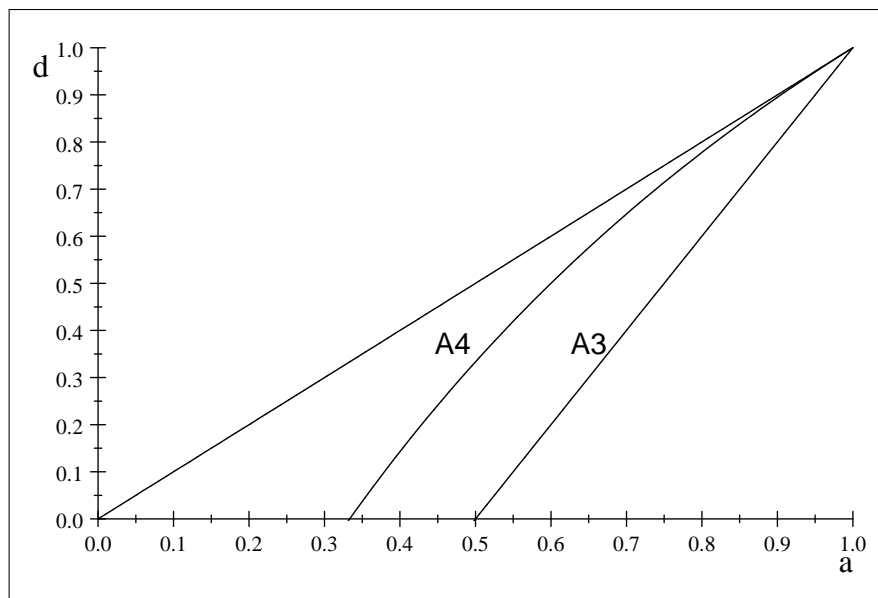
Notice that if  $(E^*, \mu)$  is a steady state equilibrium and  $\mu \in (0, 1)$  then expected number of offspring of a male does not depend on his attribute. Similarly, if  $E^* \in (0, 1)$  the female with endowment  $E^*$  is indifferent between the two markets.

Observe that the pairs  $(0, 1)$  and  $(0, 1)$  are both steady-state equilibria according to this definitions. The first one corresponds to the case where there are no males with  $d$  and each female enters the market for  $a$ . The second one corresponds to the case where each male has the attribute  $d$  and each female enters this market. The rest of this note shows that (i) there is no other equilibrium,

and (i) the equilibrium  $(0, 1)$  is unstable and the equilibrium  $(1, 0)$  is stable. That is, there is a unique stable equilibrium in which each male has the disadvantageous attribute.

In what follows we give specification for the reproduction functions.

**Example.** Let  $a, d \in (0, 1)$  and  $q(c, E) = c + (1 - c)E$ . If  $a > d$  then A1 and A2 are satisfied. In addition, if  $d > 2a - 1$  then A3 is also satisfied. Finally, A4 is satisfied whenever  $d > (3a - 1) / (a + 1)$ . In this linear case, A4 implies A3.



### 3 Equilibria

The goal of this section is to prove the following

**Proposition 3** *The only steady-state equilibria are  $(0, 1)$  and  $(1, 0)$ .*

To see that  $(0, 1)$  is an equilibrium, note that if each male has attribute  $a$  then  $\mu$  stays zero over time and the best-response of a female is to enter the  $a$ -market, that is,  $E^* = 1$ . Similarly, if each male has attribute  $d$  then  $\mu$  stays one over time and the best-response of a female is to enter the  $d$ -market, that is,  $E^* = 0$ . It remains to show that there is no interior equilibrium. In any interior equilibrium there must be more males than females in market  $a$ , and vice versa, there are more males than females in market  $d$ . For otherwise, the female with  $E^*$  would strictly prefer to enter market  $a$  because she would be matched with higher probability, and she would have more offspring conditional on being matched.

*Constant  $\mu$ .*— First, we show that whenever there are more females than males in the  $d$ -market,  $\mu$  is increasing. Suppose that  $1 - \mu < E^*$ . Note that for all  $E^* \in (0, 1)$

$$\mathbb{E}[q(d, E) : E \geq E^*] > q(a, E^*) > \mathbb{E}[q(a, E) : E \leq E^*],$$

where the first inequality follows from A3 and the second one from the monotonicity of  $q(a, \cdot)$ . The LHS is the expected number of male offspring of a  $d$ -male if he reproduces for sure, that is, whenever  $1 - \mu < E^*$ . The RHS is the expected number of offspring of an  $a$ -male conditional on being matched. Since  $1 - \mu < E^*$ , an  $a$ -male is unmatched with positive probability and hence, his expected number of offspring is even smaller than the RHS. In summary, whenever  $1 - \mu < E^*$  the fraction of  $d$ -males increases.

The following condition guarantees that the reproductive value of a male is independent of his type

$$\frac{1 - E^*}{\mu} \mathbb{E}[q(d, E) : E \geq E^*] = \mathbb{E}[q(a, E) : E \leq E^*],$$

or equivalently

$$\int_{E^*}^1 q(d, E) dE = \frac{\mu}{E^*} \int_0^{E^*} q(a, E) dE. \quad (1)$$

Define  $\mu_1(E^*)$  as the solution for the following equality:

$$\int_{E^*}^1 q(d, E) dE = \frac{\mu_1(E^*)}{E^*} \int_0^{E^*} q(a, E) dE$$

Note that, by (1),  $(\mu_1(E^*), E^*)$  are the set of those states where the reproductive value of a male does not depend on his attribute. Observe that the curve  $\mu_1$  is only defined if  $E^* \geq \tilde{E}$ , where  $\tilde{E}$  solves<sup>2</sup>

$$\int_{\tilde{E}}^1 q(d, E) dE = \frac{1}{\tilde{E}} \int_0^{\tilde{E}} q(a, E) dE.$$

The left-hand side is decreasing in  $\tilde{E}$  and the right-hand side is increasing in  $\tilde{E}$ . At  $\tilde{E} = 0$  the LHS is larger than the right-hand side by A3 ( $Eq(d, E) > q(a, 0)$ ), and at  $\tilde{E} = 1$  the RHS is larger than the LHS ( $0 < Eq(a, E)$ ). Therefore, the Intermediate Value Theorem guarantees a unique solution for the previous equality.

*Indifferent  $E^*$ .*— The following condition guarantees that  $E^*$  is indifferent between the two markets:

$$q(d, E^*) = \frac{1 - \mu}{E^*} q(a, E^*),$$

or equivalently,

$$\frac{q(d, E^*)}{q(a, E^*)} = \frac{1 - \mu}{E^*} \quad (2)$$

Define  $\mu_2(E^*)$  as the solution for the following equality:

$$q(d, E^*) = \frac{1 - \mu_2(E^*)}{E^*} q(a, E^*).$$

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<sup>2</sup>If  $E^* < \tilde{E}$ , the LHS is strictly larger than the RHS even if  $\mu = 1$ , that is, the  $d$ -males have more offspring than the  $a$ -males.

Note that, by (2),  $(\mu_1(E^*), E^*)$  are the set of those states where a female with endowment  $E^*$  is indifferent between the two markets. This curve is defined on  $[0, 1]$ .

The following lemma characterizes some properties of  $\mu_1$  and  $\mu_2$ .

**Lemma 4** (i)  $\mu_1(1) = \mu_2(1)$  and  
(ii)  $\mu_1$  and  $\mu_2$  are decreasing.

**Proof.** Trivial. ■

Suppose that  $(\mu, E^*)$  is an interior equilibrium. Then obviously  $\mu_1(E^*) = \mu_2(E^*) = \mu$ . By (1)

$$\frac{E^* \int_{E^*}^1 q(d, E) dE}{\int_0^{E^*} q(a, E) dE} = \mu$$

and, hence,

$$\frac{1}{E^*} = \frac{q(d, E^*)}{q(a, E^*)} + \frac{\int_{1-E^*}^1 q(d, E) dE}{\int_0^{E^*} q(a, E) dE}. \quad (3)$$

**Lemma 5** *There does not exist an  $E^* \in (\tilde{E}, 1)$  such that  $\mu_1(E^*) = \mu_2(E^*)$ , and hence, there is no interior equilibrium.*

For the proof of this lemma we need the following result.

**Lemma 6** *The following expression is increasing in  $x$ :*

$$\frac{E(q(a, E) : E \leq x)}{E(q(d, E) : E \geq x)}.$$

**Proof.** See the Appendix. ■

We are ready to prove Lemma 5.

**Proof of Lemma 5.** Note that  $\mu_1(\tilde{E}) = 1 < \mu_2(\tilde{E})$ . Hence, it is enough to show that there does not exist an  $E^*$  for which  $\mu_1(E^*) = \mu_2(E^*)$ . ■

**Lemma 7 Proof.** *It is enough to show that (3) does not hold if  $E^* < 1$ . By contradiction, assume that  $E^* (< 1)$  solves (3). Rewrite (3) as follows*

$$\frac{1}{E^*} = \frac{q(d, E^*) \int_0^{E^*} q(a, E) dE + q(a, E^*) \int_{1-E^*}^1 q(d, E) dE}{q(a, E^*) \left( \int_0^{E^*} q(a, E) dE \right)}.$$

By A3  $q(a, E^*) < E(q(d, E') : E' \geq E^*) = \int_{1-E^*}^1 q(d, E) dE / (1 - E^*)$ . Hence, from the previous equality,

$$\frac{1}{E^* (1 - E^*)} > \frac{q(d, E^*) \int_0^{E^*} q(a, E) dE + q(a, E^*) \int_{1-E^*}^1 q(d, E) dE}{\left( \int_{1-E^*}^1 q(d, E) dE \right) \left( \int_0^{E^*} q(a, E) dE \right)}. \quad (4)$$

The key observation is that

$$\begin{aligned} \frac{\partial \log \frac{\int_0^{E^*} q(a, E) dE}{\int_{1-E^*}^1 q(d, E) dE}}{\partial E^*} &= \frac{q(d, E^*) \int_0^{E^*} q(a, E) dE + q(a, E^*) \int_{1-E^*}^1 q(d, E) dE}{\left( \int_0^{E^*} q(a, E) dE \right) \left( \int_{1-E^*}^1 q(d, E) dE \right)} \text{ and} \\ \frac{\partial \log \frac{E^*}{1-E^*}}{\partial E^*} &= \frac{1}{E^* (1-E^*)} \end{aligned}$$

Therefore, (4) can be rewritten as

$$\frac{\partial \log \frac{E^*}{1-E^*}}{\partial E^*} > \frac{\partial \log \frac{\int_0^{E^*} q(a, E) dE}{\int_{1-E^*}^1 q(d, E) dE}}{\partial E^*}$$

Then, by continuity, there exists an interval  $(X, Y)$  ( $X, Y \in (0, 1)$ ) where the previous inequality holds. By the Fundamental Theorem of Calculus,

$$\log \frac{Y}{1-Y} - \log \frac{X}{1-X} > \log \frac{\int_0^Y q(a, E) dE}{\int_Y^1 q(d, E) dE} - \log \frac{\int_0^X q(a, E) dE}{\int_X^1 q(d, E) dE},$$

which is equivalent to

$$\log \frac{E(q(a, E) : E \leq Y)}{E(q(d, E) : E \geq Y)} < \log \frac{E(q(a, E) : E \leq X)}{E(q(d, E) : E \geq X)},$$

which is a contradiction by Lemma 6. ■

An immediate implication of the previous lemma is the following

**Corollary 8**  $\mu_1(E) > \mu_2(E)$  for all  $E \in (\tilde{E}, 1)$ .

**Proof.** Note that  $\mu_1(\tilde{E}) = 1 < \mu_2(\tilde{E})$ . This implies that if  $\mu_2(E) > \mu_1(E)$  for an  $E \in (\tilde{E}, 1)$  then the two curves would intersect because both  $\mu_1$  and  $\mu_2$  are continuous. But this is impossible by the previous lemma. ■

## 4 Dynamic Stability

We consider an abstract evolutionary dynamic process based on the following idea. If the expected number of offspring of a male with attribute  $a$  is larger than that of a male with attribute  $d$  in a given period then there the proportion of  $a$  males will increase. Similarly, if the expected number of offspring of a female with endowment  $E^*$  is larger in market  $a$  than in market  $d$  then  $E^*$  increases.

To be more specific let us assume that the state of the environment is  $(\mu_0, E_0^*)$ . The dynamic process is defined by  $(\psi, \varphi) : \mathbb{R}_+ \times [0, 1]^2 \rightarrow [0, 1]^2$ . The interpretation is that if the initial state of the environment is  $(\mu_0, E_0^*)$  then at time  $t$  the state becomes  $(\psi_t(\mu_0, E_0^*), \varphi_t(\mu_0, E_0^*))$ .

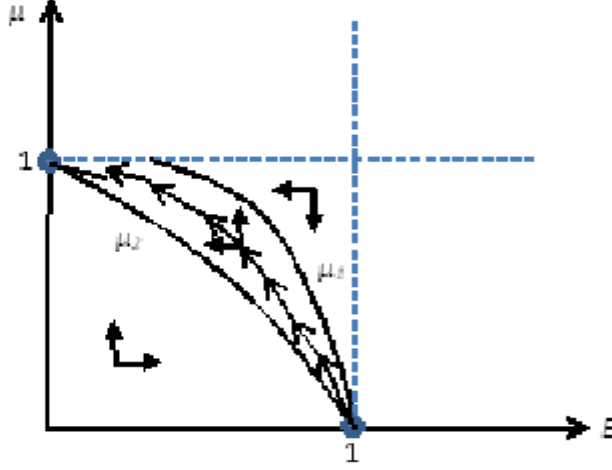


Figure 1: Phase Diagram

We require this dynamic process to satisfy two assumptions. First,  $\dot{\psi}_t(\mu_0, E_0^*) > (<) 0$  if and only if

$$\frac{1 - E^*}{\mu_t} \mathbb{E}[q(d, E) : E \geq E_t^*] > (<) \mathbb{E}[q(a, E) : E \leq E_t^*].$$

Second,  $\dot{\varphi}_t(\mu_0, E_0^*) > (<) E_0^*$  if and only if

$$q(d, E_t^*) < (>) \frac{1 - \mu}{E_t^*} q(a, E_t^*).$$

In other words, if a state  $(\mu_t, E_t^*)$  is above (below) the curve  $\mu_1$  then  $\mu_t$  decreases (increases). Similarly, if  $(\mu_t, E_t^*)$  is to the right (left) of the  $\mu_2$  curve then  $E_t^*$  decreases (increases).

**Definition 9** *The state  $(\mu, E^*)$  is a stable equilibrium if (i) it is a steady-state equilibrium, and (ii) for all  $\varepsilon > 0$  there exists an  $\delta > 0$ , such that if  $|\mu_0 - \mu|, |E_0^* - E^*| < \delta$  then*

$$\left| \psi_t(\mu_0, E_0^*) - \mu \right|, \left| \varphi_t(\mu_0, E_0^*) - E^* \right| < \varepsilon.$$

We are ready to state the main theorem of the paper.

**Theorem 10** *The state  $(1, 0)$  is the unique stable equilibrium.*

We will explain that the dynamic system defined by  $(\psi, \varphi)$  generates the Phase diagram depicted on Figure 1.

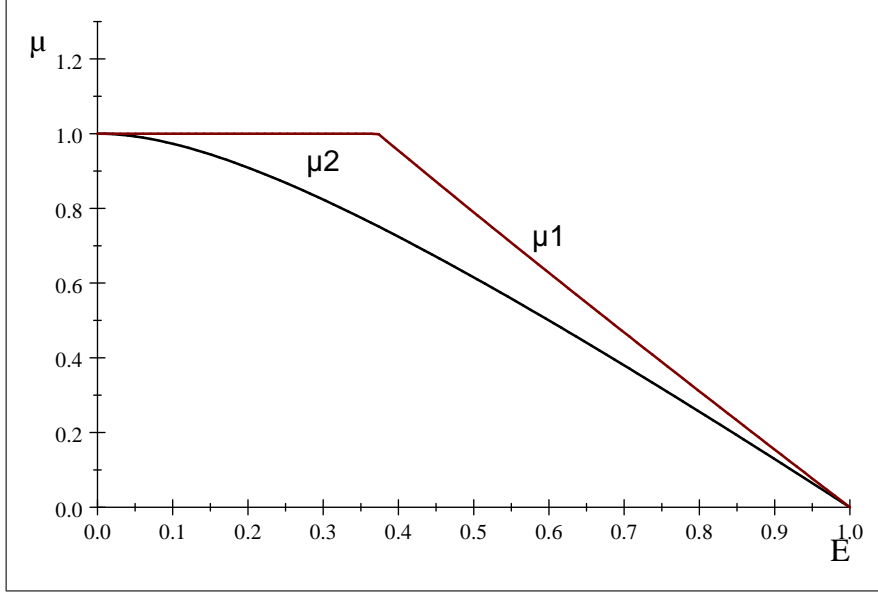
On Figure 1, the horizontal arrows show the direction of the best-response cutoff,  $E^*$ . Similarly, the vertical arrows describe the change of the fraction of  $d$ -males. Note first, that by Corollary 8, the curve  $\mu_1$  is strictly above the curve  $\mu_2$  on  $(0, 1)$ . By the definitions of  $\mu_1$  and  $\varphi$ , the vertical arrow at a state  $(\mu, E)$  points to the left if and only if  $(\mu, E)$  is above  $\mu_1$ . By the definitions of  $\mu_2$  and  $\psi$ , the vertical arrow at a state  $(\mu, E)$  points downwards if and only if  $(\mu, E)$  is above  $\mu_2$ .



**Example cont.** Let  $a = .3$  and  $d = 0$ . Then

$$\mu_1(E) = \min \left\{ 1, \frac{(1-E) \frac{(1+E)}{2}}{(.3) + (1-(.3)) \frac{E}{2}} \right\}$$

$$\mu_2(E) = 1 - \frac{E^2}{(.3) + (1-(.3)) E}$$



## 5 Appendix

**Proof of Lemma 6.** We have to show that

$$\frac{E(q(a, E) : E \leq x)}{E(q(d, E) : E \geq x)} = \frac{(1-x) \int_0^x q(a, E) dE}{x \int_x^1 q(d, E) dE}$$

is increasing in  $x$ . This is increasing if and only if

$$\begin{aligned} & \left[ (1-x)q(a, x) - \int_0^x q(a, E) dE \right] x \int_x^1 q(d, E) dE \\ - & \left[ \int_x^1 q(d, E) dE - xq(d, x) \right] (1-x) \int_0^x q(a, E) dE \\ = & x(1-x) \left[ q(a, x) \int_x^1 q(d, E) dE + q(d, x) \int_0^x q(a, E) dE \right] \\ - & \int_0^x q(a, E) dE \int_x^1 q(d, E) dE > 0. \end{aligned}$$

The inequality can be rewritten as

$$\begin{aligned}
& x(1-x)q(a,x) \int_x^1 q(d,E) dE - (1-x) \int_0^x q(a,E) dE \int_x^1 q(d,E) dE \\
&= (1-x) \int_x^1 q(d,E) dE \left[ xq(a,x) - \int_0^x q(a,E) dE \right] \\
&> x \int_0^x q(a,E) dE \int_x^1 q(d,E) dE - x(1-x)q(d,x) \int_0^x q(a,E) dE \\
&= x \int_0^x q(a,E) dE \left[ \int_x^1 q(d,E) dE - (1-x)q(d,x) \right].
\end{aligned}$$

After dividing through by  $x(1-x) \int_0^x q(a,E) dE \int_x^1 q(d,E) dE$  we get

$$\frac{1}{x} \left[ \frac{q(a,x)}{E(q(a,E) : E \leq x)} - 1 \right] > \frac{1}{1-x} \left[ 1 - \frac{q(d,x)}{E(q(d,E) : E \geq x)} \right]. \quad (5)$$

Since  $q$  is concave in  $E$ ,

$$\begin{aligned}
E(q(a,E) : E \leq x) &\leq q(a,x) - \frac{x}{2} \frac{\partial q(a,E)}{\partial E} \Big|_{E=x} \quad \text{and} \\
E(q(d,E) : E \geq x) &\leq q(d,x) + \frac{1-x}{2} \frac{\partial q(d,E)}{\partial E} \Big|_{E=x}.
\end{aligned}$$

Hence, in order to show (5) it is enough to prove that

$$\frac{1}{x} \left[ \frac{q(a,x)}{q(a,x) - \frac{x}{2} \frac{\partial q(a,E)}{\partial E} \Big|_{E=x}} - 1 \right] > \frac{1}{1-x} \left[ 1 - \frac{q(d,x)}{q(d,x) + \frac{1-x}{2} \frac{\partial q(d,E)}{\partial E} \Big|_{E=x}} \right].$$

this inequality simplifies to

$$\frac{\frac{\partial q(a,E)}{\partial E} \Big|_{E=x}}{q(a,x) - \frac{x}{2} \frac{\partial q(a,E)}{\partial E} \Big|_{E=x}} > \frac{\frac{\partial q(d,E)}{\partial E} \Big|_{E=x}}{q(d,x) + \frac{1-x}{2} \frac{\partial q(d,E)}{\partial E} \Big|_{E=x}}.$$

After multiplying through by the product of the dominators:

$$q(d,x) \frac{\partial q(a,E)}{\partial E} \Big|_{E=x} + \frac{\frac{\partial q(a,E)}{\partial E} \Big|_{E=x} \frac{\partial q(d,E)}{\partial E} \Big|_{E=x}}{2} - q(a,x) \frac{\partial q(d,E)}{\partial E} \Big|_{E=x} > 0,$$

which can be rewritten as

$$\frac{q(d,x)}{\frac{\partial q(d,E)}{\partial E} \Big|_{E=x}} + \frac{1}{2} \geq \frac{q(a,x)}{\frac{\partial q(a,E)}{\partial E} \Big|_{E=x}}.$$

This is just (A4). ■