Analyzing Fiscal Policies in a Heterogeneous-Agent Overlapping-Generations Economy*

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1 Introduction

The heterogeneous-agent overlapping-generations (OLG) model is the most persuasive framework for analyzing fiscal policy changes. Unlike the representative-agent infinite-horizon model, the heterogeneous-agent OLG model (1) accommodates the lifecycle properties that are important for determining labor supply and savings choices, (2) allows for *intra*-generational heterogeneity in households, which is necessary for analyzing the impact of policy changes on the income and wealth distributions, and (3) incorporates the foundation for the relevancy of the *inter*-generational distribution of wealth, which is important for analyzing fiscal policies that change the timing of taxes.

Solving a heterogeneous-agent OLG model that is rich enough to analyze a realistic fiscal policy change, however, can be computationally challenging. There are technically infinitely many heterogeneous agents in the model economy, and we need to solve their optimization problems for many periods. The wealth distribution must also be tracked over time. Only the simplest varieties of OLG models can be solved using "black box" large-scale constrained optimizers. Heterogeneous-agent models, such as that considered herein, must be tackled using the explicit formulation of dynamic programming that is embedded within a general optimization routine to close the system in general equilibrium.

This chapter shows how to solve a heterogeneous-agent OLG model that includes elastic labor supply as well as a progressive individual income tax and a social security system that are similar to those in the U.S. economy. We solve for the Kuhn Tucker conditions for each agent in each period by using a Newton-type nonlinear equation solver. The decision rules of a heterogeneous -agent is then embedded within a Gauss-Jacobi iteration of factor prices and government policy variables in order to produce rational expectations of those in the steady-state equilibrium and the equilibrium transition path. This approach is straightforward and fairly "general purpose" in nature, thereby allowing it to be easily extended to accommodate additional complexity.

The rest of the chapter is organized as follows: Section 2 describes the stylized heterogeneous-agent OLG model with a progressive income tax and a realistic social security system, Section 3 explains the computational algorithms to solve the optimization problem of heterogeneous agents and to solve the overall model economy for an equilibrium, Section 4 shows the calibration of the baseline economy to the U.S. economy, Section 5 demonstrates the features of the heterogeneous-agent OLG model numerically by using a simple consumption tax reform and social security privatization as examples, and Section 6 concludes the chapter.

2 The Model Economy

The economy consists of a large number of overlapping-generations households, a perfectly competitive representative firm with constant-returns-to-scale technology, and a government with a commitment technology. Households are heterogeneous and face uninsurable income risks in the tradition of Bewley (1986), Huggett (1993), Aiyagari (1994), Carroll (1997), and many others. The time is discrete and one model period is a year, which is denoted by t. In a steady-state equilibrium of the model, the economy is assumed to be on a balanced growth path with a labor-augmenting productivity growth rate   and a population growth rate . In the following model description, individual variables other than working hours are thus growth-adjusted by $(1 + \quad)^{\quad t}$ and aggregate variables are adjusted by $[\,(1 + \quad)(1 + \quad)\,]^{\quad t}$.

2.1 The Households

The households are heterogeneous with respect to the age, $i = 1, \ldots, I$, beginning-of-period wealth, $a \in A = [0, a_{\max}]$, average historical earnings, $b \in B = [0, b_{\max}]$, and working ability, $e \in E = [0, e_{\max}]$. The households enter the economy and start working at age $i = 1$, which is real age 21. They retire at age I_R and live at most up to age I. The average historical earnings are used for the average indexed monthly earnings (AIME) and determine individual Social Security benefits. The individual working ability is equivalent with an hourly wage, and it follows the first-order Markov process. In each year, t, a household receives working ability shock, e, and chooses consumption, c, working hours, h, and wealth at the beginning of the next year, a', to maximize her expected lifetime utility.

State Variables. Let $\mathbf{s}$ and $\mathbf{S}_t$ denote the individual state of a household and the aggregate state of the economy in year t, respectively,

$$\mathbf{s} = (i, a, b, e), \qquad \mathbf{S}_t = (x(\mathbf{s}), W_{G,t}),$$

where $x(\mathbf{s})$ is the population density function of households and $W_{G,t}$ is the government net wealth at the beginning of year t. Let $\mathbf{\Psi}_t$ be the government policy schedule committed at the beginning of year t,

$$\mathbf{\Psi}_t = \quad C_{G,s}; tr_{LS,s}; \quad _{I,s}(\cdot); \quad _{C,s} \quad _{P,s}(\cdot); tr_{SS,s}(\cdot); W_{G,s+1} \quad {}_{s=t}^{1};$$

where $C_{G,t}$ is government consumption, $tr_{LS,t}$ is a lump sum transfer per household, $\quad_{I,t}(\cdot)$ is a progressive income tax function, $\quad_{C,t}$ is a flat consumption tax rate, $\quad_{P,t}(\cdot)$ is a Social Security payroll tax function, $tr_{SS,t}(\cdot)$ is a Social Security benefit function, and $W_{G,t+1}$ is government net wealth at the beginning of the next year.

The Optimization Problem. Let $v(\mathbf{s}; \mathbf{S}_t; \mathbf{\Psi}_t)$ be the value function of a heterogeneous household at the beginning of year t. Then, the household's optimization problem is

$$(1) \quad v(\mathbf{s}; \mathbf{S}_t; \mathbf{\Psi}_t) = \max_{c,h,a'} \text{n} \; u(c; h) + \tilde{} \quad _iE \quad v(\mathbf{s}^0; \mathbf{S}_{t+1}; \mathbf{\Psi}_{t+1}) \,|\, \mathbf{s} \quad \text{o}$$

subject to the constraints for the decision variables,

$$(2) \quad c > 0; \qquad 0 \le h < 1; \qquad a^0 \ge 0;$$

and the law of motion of the individual state,

$$(3) \quad \mathbf{s}^0 = (i+1; a^0; b^0; e^0);$$

$$(4) \quad a^0 = \frac{1}{1+} \quad (1+r_t)a + w_t eh - \quad _{I,t}(r_t a; w_t eh) - \quad _{P,t}(w_t eh) + tr_{SS,t}(i; b) + tr_{LS,t} - (1 + \quad _{C,t})c \quad ;$$

$$(5) \quad b^0 = \mathbf{1}_{fi<I_R g} \frac{1}{i} \quad (i-1)b + \min(w_t eh; \#_{\max}) \quad + \mathbf{1}_{fi \quad I_R g} b;$$

where $u(\cdot)$ is the period utility function, $\tilde{}$ is the growth-adjusted discount factor explained below, $\quad_i$ is the conditional survival probability at the end of age i given that the household is alive at the beginning of age i, $E[\cdot]$ is the expected value operator. In the law of motion, r_t is the interest rate, w_t is the wage rate per efficiency unit of labor, $\mathbf{1}_{f\,g}$ is an indicator function that returns 1 if the condition in $\{\ \}$ holds and 0 otherwise, I_R is an exogenous retirement age, and $\#_{\max}$ is the maximum taxable earnings for the OASI program. The end-of-period wealth, a^0, is adjusted by the productivity growth rate, $1 +$. The average historical earnings, b', are wage-indexed (i.e., not growth-adjusted), but the price indexation of AIME after age 60 is reflected in the Social Security benefit function.

Perfect Annuity Markets. When perfect annuity markets are available in the model economy, the household's budget constraint, equation (4), is modified to

$$a' = \frac{1}{(1+\ \ )\ _i}\ \ (1+r_t)a + w_t eh - \ _{I,t}(r_t a; w_t eh) - \ _{P,t}(w_t eh) + tr_{SS,t}(i; b) + tr_{LS,t}$$
$$- (1+\ _{C,t})c\ :$$

In the absence of intentional bequest motive, household wealth is fully annuitized, and the actuarially fair price of annuity is $1=\ _i$.

The Household's Preference. The household's period utility function is a combination of Cobb-Douglas and constant relative risk aversion,

$$u(c; h) = \frac{c^{\alpha}(1-h)^{1\ \ \alpha\ \ 1\ \ \gamma}}{1-\ \ };$$

which is consistent with a growth economy. With this preference, the growth-adjusted discount factor is $\tilde{\ } = \ \ (1+\ \ )^{\alpha(1\ \ \gamma)}$, where   is the original discount factor. While the solution approach discussed in the later section works for other utility functions as well, it is important that any utility function satisfies the requirements of balanced growth path as derived in King, Plosser, and Rebelo (1988).

The Government's Policy Functions. The individual income tax function is one of Gouveia and Strauss (1994),

$$\ _{I,t}(y) = \ '_t\ \ \text{h}\ y - \ y\ ^{\varphi_1} + \ '_2\ \ ^{1/\varphi_1}\ \text{i}\ ;$$

where $y = \max(r_t a + w_t eh - d; 0)$ is the household's taxable income with deductions and exemptions d. While a smooth tax function is not strictly required, it does tend to speed up the rate of convergence to a fixed point without giving up much precision. The Social Security payroll tax function is

$$\ _{P,t}(w_t eh) = \bar{\ }_{P,t} \min(w_t eh; \#_{\max});$$

where $^{-}{}_{P,t}$ is the flat Old-Age and Survivors Insurance (OASI) tax rate that includes the employer portion. The payroll tax ceiling is captured by $_{\max}$ and produces a non-convexity in the households optimization problem. Finally, the Social Security benefit function is kinked and equal to:

$$
\begin{aligned}
tr_{SS,t}(i; b) = \mathbf{1}_{\{i \quad I_R\}} \quad {}_t(1 + \quad)^{40 \quad i} \quad & 0.90 \min(b_j; \#_1) \\
& + 0.32 \max\left[\min(b_j; \#_2) - \#_1; 0\right] + 0.15 \max(b_j - \#_2; 0) \quad ;
\end{aligned}
$$

where $\#_1$ and $\#_2$ are the thresholds for the 3 replacement rate brackets, 90%, 32%, and 15%, that calculate the social security benefit from the average historical earnings, and $_t$ is the benefit adjustment factor to balance the budget.

Decision Rules. Solving the household's problem for c, h, and a' for all possible states, we obtain the household's decision rules and the average historical earnings in the next period as $c(\mathbf{s}; \mathbf{S}_t; \mathbf{\Psi}_t)$, $h(\mathbf{s}; \mathbf{S}_t; \mathbf{\Psi}_t)$, and

$$
\begin{aligned}
a'(\mathbf{s}; \mathbf{S}_t; \mathbf{\Psi}_t) = \frac{1}{1+} \quad & (1 + r_t)a + w_t e h(\mathbf{s}; \mathbf{S}_t; \mathbf{\Psi}_t) - \quad {}_{I,t}(r_t a; w_t e h(\mathbf{s}; \mathbf{S}_t; \mathbf{\Psi}_t)) \\
& - \quad {}_{P,t}(w_t e h(\mathbf{s}; \mathbf{S}_t; \mathbf{\Psi}_t)) + tr_{SS,t}(i; b) + tr_{LS,t} - (1 + \quad {}_{C,t})\, c(\mathbf{s}; \mathbf{S}_t; \mathbf{\Psi}_t) \quad ; \\
b'(\mathbf{s}; \mathbf{S}_t; \mathbf{\Psi}_t) = \mathbf{1}_{\{i < I_R\}} \frac{1}{i} \quad & (i - 1)b_1 + \min(w_t e h(\mathbf{s}; \mathbf{S}_t; \mathbf{\Psi}_t); \#_{\max}) \quad + \mathbf{1}_{\{i \quad I_R\}} b.
\end{aligned}
$$

The Distribution of Households. Let $x_t(\mathbf{s})$ be the population density function of households in period t, and let $X_t(\mathbf{s})$ be the corresponding cumulative distribution function. We assume that households enter the economy with no assets and working histories, i.e., $a = b = 0$, and that the growth-adjusted population of age $i = 1$ households is normalized to unity,

$$
\int_{A \quad B \quad E} dX_t(1; a; b; e) = \int_E dX_t(1; 0; 0; e) = 1.
$$

The law of motion of the growth-adjusted population distribution is

$$
x_{t+1}(\mathbf{s}') = \frac{1}{1+} \int_{A \quad B \quad E} \mathbf{1}_{\{a'=a'(\mathbf{s},\mathbf{S}_t;\mathbf{\Psi}_t),\, b'=b'(\mathbf{s},\mathbf{S}_t;\mathbf{\Psi}_t)\}} \quad {}_i(e' \,|\, e)\, dX_t(\mathbf{s});
$$

where is the population growth rate, and $_i(e^0 \mid e)$ is the transition probability density function of working ability.

Aggregation. Total private wealth, $W_{P,t}$, national wealth, K_t, which is equal to capital stock in a closed economy, and labor supply in efficiency units, L_t, are

$$(6) \quad W_{P,t} = \sum_{i=1}^{I} \int_{A \quad B \quad E} a\, dX_t(\mathbf{s}); \qquad K_t = W_{P,t} + W_{G,t};$$

$$(7) \quad L_t = \sum_{i=1}^{I_R \; 1} \int_{A \quad B \quad E} eh(\mathbf{s}; \mathbf{S}_t; \mathbf{\Psi}_t)\, dX_t(\mathbf{s}):$$

2.2 The Firm

In each period, the representative firm chooses the capital input, $\tilde{K}_t$, and efficiency labor input, $\tilde{L}_t$, to maximize its profit, taking factor prices, r_t and w_t, as given, i.e.,

$$(8) \quad \max_{\tilde{K}_t, \tilde{L}_t} F(\tilde{K}_t; \tilde{L}_t) - (r_t + \quad)\tilde{K}_t - w_t \tilde{L}_t;$$

where $F(\cdot)$ is a constant-returns-to-scale production function,

$$F(\tilde{K}_t; \tilde{L}_t) = A\tilde{K}_t^{\theta}\tilde{L}_t^{1 \quad \theta};$$

with total factor productivity A, and is the depreciation rate of capital. The profit maximizing conditions are

$$(9) \quad F_K(\tilde{K}_t; \tilde{L}_t) = r_t + \quad ; \qquad F_L(\tilde{K}_t; \tilde{L}_t) = w_t;$$

and the factor markets clear when $K_t = \tilde{K}_t$ and $L_t = \tilde{L}_t$.

A Small Open Economy. In a small open economy, factor prices, r_t and w_t, are fixed at baseline (international) levels, as international capital flows ensure that the capital-labor ratio determined by the world interest rate is attained in the economy. In an equilibrium, the domestic capital stock, $K_{D,t}$, and labor supply,

L_t, therefore, are determined so that the firm's profit maximizing condition satisfies

$$F_K(K_{D,t}, L_t) = r_t + \; , \qquad F_L(K_{D,t}, L_t) = w_t.$$

The gross domestic product, $Y_{D,t}$, is determined by the production function,

$$Y_{D,t} = F_K(K_{D,t}, L_t),$$

and gross national product, Y_t, is determined by

$$Y_t = (r_t + \;)(W_{P,t} + W_{G,t}) + w_t L_t.$$

2.3 The Government

The government's income tax revenue is

$$(10) \quad T_{I,t}(\;_t) = \sum_{i=1} \int_{A \; B \; E} \;_{I,t}(r_t a + w_t e h(\mathbf{s}, \mathbf{S}_t, \mathbf{\Psi}_t); \;_t)\, dX_t(\mathbf{s}),$$

and the consumption tax revenue is

$$(11) \quad T_{C,t}(\;_{C,t}) = \sum_{i=1} \int_{A \; B \; E} \;_{C,t}\, c(\mathbf{s}, \mathbf{S}_t, \mathbf{\Psi}_t)\, dX_t(\mathbf{s}).$$

For simplicity, we assume that the government collects remaining wealth held by deceased households at the end of period t and distributes it in a lump-sum manner in the same period.[1] Since there are no aggregate shocks in the model economy, the government can perfectly predict the sum of accidental bequests during the period. The government revenue from these accidental bequests is

$$(12) \quad BQ_t = \sum_{i=1} \int_{A \; B \; E} (1 - \;_i)(1 + \;) a'(\mathbf{s}, \mathbf{S}_t, \mathbf{\Psi}_t)\, dX_t(\mathbf{s}).$$

If there are not any additional lump-sum transfers, $TR_{LS,t} = BQ_t$ holds, and the individual lump-sum

[1] Nishiyama and Smetters (2007) have considered a little more realistic distribution methods that better preserve the distribution of wealth.

transfer is calculated as

$$(13)\quad tr_{LS,t} = \left(\sum_{i=1} \int_{A \quad B \quad E} dX_t(\mathbf{s}) \right)^{\ 1} BQ_t.$$

Otherwise, the aggregate transfers are obtained as

$$(14)\quad TR_{LS,t}(tr_{LS,t}) = \sum_{i=1} \int_{A \quad B \quad E} tr_{LS,t}\, dX_t(\mathbf{s}).$$

The government's OASI payroll tax revenue, $T_{P,t}$, is

$$(15)\quad T_{P,t}(\bar{\ }_{P,t}) = \sum_{i=1}^{I_R \ 1} \int_{A \quad B \quad E} {}_{P,t}(w_t eh(\mathbf{s};\mathbf{S}_t;\mathbf{\Psi}_t);\bar{\ }_{P,t})\, dX_t(\mathbf{s});$$

and the OASI benefit expenditure, $TR_{SS,t}$, is

$$(16)\quad TR_{SS,t}(\ _t) = \sum_{i=I_R} \int_{A \quad B \quad E} tr_{SS,t}(i;b_1;b_2;m;\ _t)\, dX_t(\mathbf{s}).$$

The law of motion of the government net wealth is

$$(17)\quad W_{G,t+1} = \frac{1}{(1+\ )(1+\ )} \Big((1+r_t)W_{G,t} + T_{I,t}(\ '_t) + T_{C,t}(\ _{C,t}) + BQ_t - C_{G,t}$$
$$- TR_{LS,t}(tr_{LS,t}) + T_{P,t}(\bar{\ }_{P,t}) - TR_{SS,t}(\ _t) \Big).$$

2.4 Recursive Competitive Equilibrium

The recursive competitive equilibrium of this model economy is defined as follows.

DEFINITION Recursive Competitive Equilibrium: Let $\mathbf{s} = (i;a;b;e)$ be the individual state of households, let $\mathbf{S}_t = (x(\mathbf{s});W_{G,t})$ be the state of the economy, and let $\mathbf{\Psi}_t$ be the government policy schedule committed at the beginning of period t,

$$\mathbf{\Psi}_t = \Big\{ C_{G,s};tr_{LS,s};\ _{I,s}(\cdot);\ _{C,s};\ _{P,s}(\cdot);tr_{SS,s}(\cdot);W_{G,s+1} \Big\}_{s=t}^{1}.$$

A time series of factor prices and the government policy variables,

$$\mathbf{\Omega}_t = \left\{ r_s, w_s, C_{G,s}, tr_{LS,s}, \varphi_s, \tau_{C,s}, \bar{\tau}_{P,s}, \theta_s, W_{G,s} \right\}_{s=t}^{\infty},$$

the value functions of households, $\{v(\mathbf{s}; \mathbf{S}_s; \mathbf{\Psi}_s)\}_{s=t}^{\infty}$; the decision rules of households,

$$\mathbf{d}(\mathbf{s}; \mathbf{S}_s; \mathbf{\Psi}_s) = \left\{ c(\mathbf{s}; \mathbf{S}_s; \mathbf{\Psi}_s), h(\mathbf{s}; \mathbf{S}_s; \mathbf{\Psi}_s), a'(\mathbf{s}; \mathbf{S}_s; \mathbf{\Psi}_s), b'(\mathbf{s}; \mathbf{S}_s; \mathbf{\Psi}_s) \right\}_{s=t}^{\infty},$$

and the distribution of households, $\{x_s(\mathbf{s})\}_{s=t}^{\infty}$, are in a recursive competitive equilibrium if, for all $s = t, \ldots, \infty$, each household solves the optimization problem (1) - (5), taking $\mathbf{S}_s$ and $\mathbf{\Psi}_s$ as given; the firm solves its profit maximization problem (8) - (9); the government policy schedule satisfies (15) - (17); and the goods and factor markets clear (6) - (7). The economy is in a steady-state equilibrium, thus on the balanced growth path, if in addition, $\mathbf{S}_s = \mathbf{S}_{s+1}$ and $\mathbf{\Psi}_{s+1} = \mathbf{\Psi}_s$ for all $s = t, \ldots, \infty$.

2.5 Welfare Measure

Suppose that the economy is in the initial steady-state equilibrium in period $t = 0$ and that the government introduces a new policy at the beginning of period 1.

Then, welfare gains or losses of newborn (age $i = 1$) households at the beginning of $t = 1, \ldots, \infty$ are calculated by the uniform percent changes, $\Delta_{1,t}$, in the baseline consumption path that would make their expected lifetime utility equivalent with the expected utility after the policy change, that is,

$$\Delta_{1,t} = \left[\left(\frac{E\, v(\mathbf{s}_1; \mathbf{S}_t; \mathbf{\Psi}_t)}{E\, v(\mathbf{s}_1; \mathbf{S}_0; \mathbf{\Psi}_0)} \right)^{\frac{1}{(1-\gamma)}} - 1 \right] \times 100.$$

Similarly, the average welfare changes of households of age i at the time of policy change ($t = 1$) are calculated by the uniform percent changes, $\Delta_{i,1}$, required in the baseline consumption path so that the rest of the lifetime value would be equal to the rest of the lifetime value after the policy change, that is,

$$\Delta_{i,1} = \left[\left(\frac{E\, v(\mathbf{s}_i; \mathbf{S}_1; \mathbf{\Psi}_1)}{E\, v(\mathbf{s}_i; \mathbf{S}_0; \mathbf{\Psi}_0)} \right)^{\frac{1}{(1-\gamma)}} - 1 \right] \times 100.$$

Note that $\Delta_{i,1}$ for $i = I, \ldots, 1$ shows the cohort-average welfare changes of all current households alive at the time of policy change, and $\Delta_{1,t}$ for $t = 2, \ldots, \infty$ shows the cohort-average welfare changes of all future households. These measures are sometimes called the "equivalent variation" of the fiscal policy change.

3 Computational Algorithm

We solve the household's optimization problem recursively from age $i = I$ to age $i = 1$ by discretizing the asset space, $A = [0, a_{\max}]$, into J nodes, $\hat{A} = \{a_1, a_2, \ldots, a_J\}$, the average historical earning space, $B = [0, b_{\max}]$, into K nodes, $\hat{B} = \{b_1, b_2, \ldots, b_K\}$, and the working ability space, $E = [0, e_{\max}]$, into L nodes for each age, $\hat{E}^i = \{e_1^i, e_2^i, \ldots, e_L^i\}$. Since the working ability of a retired household is assumed to be $e_1^i = 0$, total number of nodes for which we solve the household's optimization problem in each period t is $(I_R - 1)JKL + (I - I_R + 1)JK$.

Let $\mathbf{\Omega}_t$ be a time series of vectors of factor prices and government policy variables that describes a future path of the aggregate economy,

$$\mathbf{\Omega}_t = \left\{ r_s, w_s, C_{G,s}, tr_{LS,s}, \varphi_s, \tau_{C,s}, \bar{\tau}_{P,s}, \psi_s, W_{G,s} \right\}_{s=t}^{\infty}.$$

The household's value function is shown as $v(\mathbf{s}, \mathbf{S}_t; \mathbf{\Psi}_t)$, and the factor prices and endogenous government policy variables are shown as $r_s(\mathbf{S}_s; \mathbf{\Psi}_s)$, $w_s(\mathbf{S}_s; \mathbf{\Psi}_s)$, $\psi_s(\mathbf{S}_s; \mathbf{\Psi}_s)$, and so on, for $s \geq t$. It is impossible to solve the model of this form because the dimension of $\mathbf{S}_t = (x(\mathbf{s}), W_{G,t})$ is infinite. In the absence of aggregate productivity or policy shocks, however, we can avoid this curse of dimensionality by replacing $(\mathbf{S}_t, \mathbf{\Psi}_t)$ with $\mathbf{\Omega}_t$. Since the time series $\mathbf{\Omega}_t$ is deterministic and perfectly foreseeable, it will suffice to find the fixed point of $\mathbf{\Omega}_t$ to solve the model economy for an equilibrium transition path.

In this section, we first explain the algorithm to solve the household's optimization problem for each individual state node,

$$\mathbf{s} = (i, a, b, e) \in \{1, 2, \ldots, I\} \times \hat{A} \times \hat{B} \times \hat{E}_i,$$

taking $\mathbf{\Omega}_t$ as given. Then, we explain how to solve the model for a steady-state equilibrium (balanced growth path) and an equilibrium transition path.

3.1 Algorithm to Solve the Household Problem

We solve the household's optimization problem backward from $i = I$ to $i = 1$ by assuming the terminal value to be zero,

$$v(\mathbf{s}_{I+1}; \mathbf{\Omega}_{t+1}) = 0 \quad \Longrightarrow \quad v_a(\mathbf{s}_{I+1}; \mathbf{\Omega}_{t+1}) = v_b(\mathbf{s}_{I+1}; \mathbf{\Omega}_{t+1}) = 0,$$

where $\mathbf{s}_i$ is the individual state vector of a household of age i. The following computational algorithm is a modified version of that in Nishiyama (2009, 2010).[2]

The Household's Problem. The optimization problem of a household at age i in period t, equations (1) - (5), is modified to

$$v(\mathbf{s}; \mathbf{\Omega}_t) = \max_{c,l} \left\{ u(c, l) + \tilde{\ }\ _iE \ v(\mathbf{s}'; \mathbf{\Omega}_{t+1}) \ \mathbf{s} \right\}$$

subject to the constraints for the decision variables,

$$0 < c \leq c_{\max}, \qquad 0 < l = 1 - h \leq 1 = l_{\max},$$

and the law of motion of the state variables,

$$\mathbf{s}' = (i + 1, a', b', e'),$$

$$a' = \frac{1}{1 +} \ (1 + r_t)a + w_t eh - \ _{I,t}(r_t a, w_t eh) - \ _{P,t}(w_t eh) + tr_{SS,t}(i, b) + tr_{LS,t} - (1 + \ _{C,t})c$$

$$= \frac{1}{1 +} \ (1 + \ _{C,t})(c_{\max} - c) \ ,$$

$$b' = \mathbf{1}_{\{i < I_R\}} \frac{1}{i} \ (i - 1)b_1 + \min(w_t e(1 - l), \#_{\max}) \ + \mathbf{1}_{\{i \ \ I_R\}} b,$$

The Complementarity Problem. Let the objective function be

$$f(c, l, \mathbf{s}, \mathbf{\Omega}_t) = u(c, l) + \tilde{\ }\ _iE \ v(\mathbf{s}'; \mathbf{\Omega}_{t+1}) \ \mathbf{s} \ .$$

[2] Auerbach and Kotlikoff (1987) solve their household's optimization problem cohort by cohort. In the presence of idiosyncratic wage shocks, however, we need to solve the problem period by period.

Then, the first-order conditions for an interior solution are

$$(18)\quad f_c(c,l;\mathbf{s};\boldsymbol{\Omega}_t) = u_c(c,l) - \frac{\tilde{\beta}_i(1+\tau_{C,t})}{1+\mu} E\left[v_a(\mathbf{s}';\boldsymbol{\Omega}_{t+1}) \mid \mathbf{s} \right] = 0,$$

$$(19)\quad f_l(c,l;\mathbf{s};\boldsymbol{\Omega}_t) = u_l(c,l) - w_t e \left[1 - \tau_{I,2,t}\left(r_t a, w_t e(1-l)\right) - \tau'_{P,t}\left(w_t e(1-l)\right) \right] \frac{u_c(c,l)}{1+\tau_{C,t}}$$

$$- \mathbf{1}_{\{i<I_R,\, w_t e(1-l)<\vartheta_{\max}\}} \frac{w_t e}{i} \tilde{\beta}_i E\left[v_b(\mathbf{s}';\boldsymbol{\Omega}_{t+1}) \mid \mathbf{s} \right] = 0,$$

where $\tau_{I,2,t}\left(r_t a, w_t e(1-l)\right)$ is the marginal labor income tax rate, and $\tau'_{P,t}\left(w_t e(1-l)\right)$ is the marginal payroll tax rate. Equation (18) is the Euler equation, and equations (19) is the marginal rate of substitution conditions of consumption for leisure.[3]

With the inequality constraints for the decision variables, the Kuhn-Tucker conditions of the household's problem are expressed as the following nonlinear complementarity problem,

$$f_c(c,l;\mathbf{s};\boldsymbol{\Omega}_t) = 0 \quad \text{if} \quad 0 < c < c_{\max}, \qquad > 0 \quad \text{if} \quad c = c_{\max},$$

$$f_l(c,l;\mathbf{s};\boldsymbol{\Omega}_t) = 0 \quad \text{if} \quad 0 < l < l_{\max}, \qquad > 0 \quad \text{if} \quad l = l_{\max},$$

which are expressed more compactly as the nonlinear system of equations,

$$(20)\quad CP(c,l) = \min\left\{ \max\left[\begin{pmatrix} f_c(c,l;\mathbf{s};\boldsymbol{\Omega}_t) \\ f_l(c,l;\mathbf{s};\boldsymbol{\Omega}_t) \end{pmatrix}, \begin{pmatrix} \varepsilon - c \\ \varepsilon - l \end{pmatrix} \right], \begin{pmatrix} c_{\max} - c \\ l_{\max} - l \end{pmatrix} \right\} = \mathbf{0},$$

where ε is a small positive number, *e.g.*, $\varepsilon = 10^{-3}$. Following Miranda and Fackler (2002), we replace the $\min(u,v)$ and $\max(u,v)$ operators with

$$\phi^-(u,v) \equiv u + v - \sqrt{u^2+v^2}, \qquad \phi^+(u,v) \equiv u + v + \sqrt{u^2+v^2},$$

respectively, to make the above system of equations differentiable without altering the solutions.

We solve equation (20) for $c(\mathbf{s};\boldsymbol{\Omega}_t)$ and $l(\mathbf{s};\boldsymbol{\Omega}_t)$ by using a Newton-type nonlinear equation solver, NEQNF, of the IMSL Fortran Numerical Library. The library function uses a modified Powell hybrid algorithm and a finite-difference approximation to the Jacobian. We evaluate the marginal values, $v_a(\mathbf{s}';\boldsymbol{\Omega}_{t+1})$

[3] The marginal utilities and marginal values in (18)-(19) can be very large and very small, depending on the state of the household. However, it is ideal to solve the problem for all households by using the same calculation. An appropriate scale adjustment of the marginal utilities and marginal values will help reducing the computation errors in a nonlinear equation solver.

and $v_b(\mathbf{s}', \boldsymbol{\Omega}_{t+1})$, between nodes in equations (18) - (19) by using either bilinear interpolation or 2-dimensional quadratic interpolation, QD2VL, of corresponding marginal value functions, equations (22) - (23), explained below.

Value and Marginal Value Functions. Once we obtain the optimal decision, we next calculate the value of the household with the current state $\mathbf{s}$ in period t as

(21) $$v(\mathbf{s}, \boldsymbol{\Omega}_t) = u\left(c(\mathbf{s}, \boldsymbol{\Omega}_t), l(\mathbf{s}, \boldsymbol{\Omega}_t)\right) + \tilde{\beta}\phi_i E\left[v(\mathbf{s}', \boldsymbol{\Omega}_{t+1}) \mid \mathbf{s}\right],$$

and the corresponding marginal values as

(22) $$v_a(\mathbf{s}, \boldsymbol{\Omega}_t) = \left\{1 + r_t\left(1 - \tau'_{I,1,t}\left(r_t a, w_t e(1 - l(\mathbf{s}, \boldsymbol{\Omega}_t))\right)\right)\right\}\frac{1}{1+\tau_{C,t}} u_c(c(\mathbf{s}, \boldsymbol{\Omega}_t), l(\mathbf{s}, \boldsymbol{\Omega}_t)),$$

(23) $$v_b(\mathbf{s}, \boldsymbol{\Omega}_t) = \frac{tr'_{SS,t}(i, b)}{1+\tau_{C,t}} u_c(c(\mathbf{s}, \boldsymbol{\Omega}_t), l(\mathbf{s}, \boldsymbol{\Omega}_t)) + \left(\mathbf{1}_{\{i<I_R\}}\frac{i-1}{i} + \mathbf{1}_{\{i \geq I_R\}}\right)\tilde{\beta}\phi_i E\left[v_b(\mathbf{s}', \boldsymbol{\Omega}_{t+1}) \mid \mathbf{s}\right],$$

where $\tau'_{I,1,t}\left(r_t a, w_t e(1 - l)\right)$ is the marginal capital income tax rate, and $tr'_{SS,t}(i, b)$ is the marginal OASI benefits with respect to b,

$$tr'_{SS,,t}(i, b) = \mathbf{1}_{\{i \geq I_R\}}\mu_t(1+\pi)^{40-i}\left(\mathbf{1}_{\{b_j<\vartheta_1\}}0.90 + \mathbf{1}_{\{\vartheta_1 \leq b_j<\vartheta_2\}}0.32 + \mathbf{1}_{\{\vartheta_2 \leq b_j\}}0.15\right).$$

3.2 Algorithm to Find the Distribution of Households

Let $x_t(\mathbf{s}) = x_t(i, a, b, e)$ be the discrete population distribution function of households in period t, where the population of age $i = 1$ households is normalized to unity,

$$\sum_{l=1}^{L} x_t(1, 0, 0, e_l) = 1.$$

Then, the law of motion of growth-adjusted population distribution is

$$x_{t+1}(i+1, a', b', e_{l'}) = \frac{1}{1+\nu}\sum_{i=1}^{I-1}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{l=1}^{L}\mathbf{1}_{\{a'=a'(\mathbf{s};\boldsymbol{\Omega}_t),\, b'=b'(\mathbf{s};\boldsymbol{\Omega}_t)\}}\pi_i(e_{l'} \mid e_l)x_t(i, a_j, b_k, e_l).$$

Since $a'(\mathbf{s}, \boldsymbol{\Omega}_t)$ and $b'(\mathbf{s}, \boldsymbol{\Omega}_t)$ are in general not on a node in $\hat{A} \times \hat{B}$, the population in the next period is linearly into 4 adjacent nodes, (a_j, b_k), (a_j, b_{k+1}), (a_{j+1}, b_k), and (a_{j+1}, b_{k+1}) such that $a_j \leq a' < a_{j+1}$

and $b_k \leq b' < b_{k+1}$.

The algorithm to calculate $x_{t+1}(\mathbf{s}')$ from $x_t(\mathbf{s})$ is as follows: First, set $x_{t+1}(\mathbf{s}') = 0$ for all $\mathbf{s}'$ and set $x_{t+1}(1, 0, 0, e_l) = \pi_1(e_l)$ for $l = 1, \ldots, L$, where $\pi_1(e_l)$ is the working ability distribution of age $i = 1$ households. Then, for $i = 1, \ldots, I$, $j = 1, \ldots, J$, $k = 1, \ldots, K$, and $l = 1, \ldots, L$, do the following:

1. Find the indexes j' and k' that satisfy $a_{j'} \leq a'(i, a_j, b_k, e_l; \mathbf{\Omega}_t) < a_{j'+1}$ and $b_{k'} \leq b'(i, a_j, b_k, e_l; \mathbf{\Omega}_t) < b_{k'+1}$.
2. Calculate the interpolation weights, $\omega_a = \dfrac{a'(i, a_j, b_k, e_l; \mathbf{\Omega}_t) - a_{j'}}{a_{j'+1} - a_{j'}}$ and $\omega_b = \dfrac{b'(i, a_j, b_k, e_l; \mathbf{\Omega}_t) - b_{k'}}{b_{k'+1} - b_{k'}}$.
3. For $l' = 1, \ldots, L$, update the next period distribution as
$$\begin{aligned} x_{t+1}(i+1, a_{j'}\!:\!a_{j'+1}, b_{k'}\!:\!b_{k'+1}, e_{l'}) &:= x_{t+1}(i+1, a_{j'}\!:\!a_{j'+1}, b_{k'}\!:\!b_{k'+1}, e_{l'}) \\ &+ \begin{bmatrix} (1-\omega_a)(1-\omega_b) & (1-\omega_a)\,\omega_b \\ \omega_a(1-\omega_b) & \omega_a\,\omega_b \end{bmatrix} \pi(e_{l'}|e_l)\, x_t(i, a_j, b_k, e_l), \end{aligned}$$
where $x_{t+1}(i+1, a_{j'}\!:\!a_{j'+1}, b_{k'}\!:\!b_{k'+1}, e_{l'})$ is a 2×2 matrix.

We can find the steady-state distribution of households by replacing $x_{t+1}(\cdot)$ with $x_t(\cdot)$ in Step 3 and calculate $x_t(i, \cdot)$ recursively from age $i = 1$ to I.[4]

3.3 Algorithm to Solve the Model for an Equilibrium

The following computational algorithm is a modified version of that in Nishiyama and Smetters (2003, 2005, 2007). The procedure to find an equilibrium transition path is shown in Conesa and Krueger (1999), followed by Domeij and Heathcote (2004) and others.

A Steady-State Equilibrium. The steady-state equilibrium with a time-invariant government policy $\mathbf{\Psi} = \{C_G, tr_{LS}, \tau_I(\cdot), \tau_C, \tau_P(\cdot), tr_{SS}(\cdot), W_G\}$ is obtained as follows:

1. Set the initial values of factor prices and government's policy variables, $\mathbf{\Omega}^0 = \{r^0, w^0, C_G^0, tr_{LS}^0, \varphi^0, \tau_C^0, \bar{\tau}_P^0, \theta^0, W_G^0\}$.
2. Given $\mathbf{\Omega}^0$, find the decision rules, $\mathbf{d}(\mathbf{s}_i; \mathbf{\Omega}^0)$, value function, $v(\mathbf{s}_i; \mathbf{\Omega}^0)$, and marginal value functions, $v_a(\mathbf{s}_i; \mathbf{\Omega}^0)$ and $v_b(\mathbf{s}_i; \mathbf{\Omega}^0)$, of a household recursively from age $i = I$ to age 1, starting with

[4] To preserve the measure of households, the distribution of households in the next period must be calculated with linear or bilinear interpolation. Yet, this procedure is much more efficient than finding the distribution by simulation.

$v(\mathbf{s}_{I+1}; \mathbf{\Omega}^0) = v_a(\mathbf{s}_{I+1}; \mathbf{\Omega}^0) = v_b(\mathbf{s}_{I+1}; \mathbf{\Omega}^0) = 0$.[5]

3. Find the steady-state population distribution of households, $x(\mathbf{s}_i)$ recursively from age $i = 1$ to age I by using the obtained decision rules, $a^0(\mathbf{s}_i; \mathbf{\Omega}^0)$ and $b^0(\mathbf{s}_i; \mathbf{\Omega}^0)$, as well as the Markov transition matrix of the working ability shock.

4. Compute K, L, and new factor prices, $(r^1; w^1)$, by using the decision rules and the population distribution function. Then, compute new government policy variables, $(C^1_G; tr^1_{LS}; \ '^1; \ ^1_C; \ ^{-1}_P; \ ^1; W^1_G)$, that satisfy the government budget constraint.

5. If the difference between $\mathbf{\Omega}^1 = \{r^1; w^1; C^1_G; tr^1_{LS}; \ '^1; \ ^1_C; \ ^{-1}_P; \ ^1; W^1_G\}$ and $\mathbf{\Omega}^0$ is small enough, then stop. Otherwise, update $\mathbf{\Omega}^0$ by using $\mathbf{\Omega}^1$ and return to Step 2.[6]

In many cases, only one or two government policy variables are endogenous and the others are exogenous. In Step 5, it will suffice to find the convergence of $(K{=}L)^0$ instead of $(r^0; w^0)$, but we usually need to dampen the iteration process of $K{=}L$ as

$$(K{=}L)^1 := \ (K{=}L)^1 + (1 - \ )(K{=}L)^0; \qquad \ \in (0; 1):$$

An Equilibrium Transition Path. Assume that the economy is in the initial steady-state equilibrium with a government policy schedule $\mathbf{\Psi}_0$ in period $t = 0$ and that the government introduces a new policy schedule $\mathbf{\Psi}_1$ at the beginning of period 1. The equilibrium transition path from the initial steady state to a new final steady-state is computed as follows:

1. Choose a large number T such that the economy is said to reach the new steady-state equilibrium within T periods. Then, set the initial values of factor prices and government's policy variables, $\mathbf{\Omega}^0_1 = \{r^0_t; w^0_t; C^0_{G,t}; tr^0_{LS,t}; \ '^0_t; \ ^0_{C,t}; \ ^{-0}_{P,t}; \ ^0_t; W^0_{G,t}\}^T_{t=1}$, that are consistent with the new policy $\mathbf{\Psi}_1$.

2. Given $\mathbf{\Omega}^0_T$, compute the final steady-state equilibrium in period T, i.e., find the decision rules, $\mathbf{d}(\mathbf{s}_i; \mathbf{\Omega}^0_T)$, value function, $v(\mathbf{s}_i; \mathbf{\Omega}^0_T)$, and marginal value functions, $v_a(\mathbf{s}_i; \mathbf{\Omega}^0_T)$ and $v_b(\mathbf{s}_i; \mathbf{\Omega}^0_T)$, of a household from age $i = I$ to age 1.

[5] Within a given age i, a household's problem at any state can be solved independently of the other states, thereby creating a large opportunity for parallelizing the computations, which is especially useful if more state variables are added.

[6] A simple Gauss-Jacobi type iteration of factor prices and government policy variables, , is more efficient than a Newton-type iteration, because the household decision rules are sensitive to the changes in .

3. Given $\mathbf{\Omega}_1^0$, find the decision rules, $\mathbf{d}(\mathbf{s}_i; \mathbf{\Omega}_t^0)$, value function, $v(\mathbf{s}_i; \mathbf{\Omega}_t^0)$, and marginal value functions, $v_a(\mathbf{s}_i; \mathbf{\Omega}_t^0)$ and $v_b(\mathbf{s}_i; \mathbf{\Omega}_t^0)$, of a household from period $t = T - 1$ to period 1, using $v(\mathbf{s}_{i+1}; \mathbf{\Omega}_{t+1}^0)$ $v_a(\mathbf{s}_{i+1}; \mathbf{\Omega}_{t+1}^0)$, and $v_b(\mathbf{s}_{i+1}; \mathbf{\Omega}_{t+1}^0)$ recursively.[7]

4. Set $x_1(\mathbf{s}) = x_0(\mathbf{s})$ and $W_{G,1}^1 = W_{G,0}$, since the economy is still in the initial steady-state equilibrium at the beginning of period $t = 1$. Compute aggregate variables, $(K_t; L_t)$, factor prices $(r_t^1; w_t^1)$, government policy variables, $(C_{G,t}^1; tr_{LS,t}^1; '^1_t; \ ^1_{C,t}; \ ^{-1}_{P,t}; \ ^1_t; W_{G,t}^1)$, and the distribution function of households, $x_{t+1}(\mathbf{s})$, recursively from period $t = 1$ to T.

5. If the difference between $\mathbf{\Omega}_1^1 = \{r_t^1; w_t^1; C_{G,t}^1; tr_{LS,t}^1; '^1_t; \ ^1_{C,t}; \ ^{-1}_{P,t}; \ ^1_t; W_{G,t}^1\}_{t=1}^T$ and $\mathbf{\Omega}_1^0$ is small enough, then stop. Otherwise, update Ω_1^0 by using Ω_1^1 and return to Step 2. If there is no change in $W_{G,T}^0$, then, return to Step 3.

If the policy change is deficit financing for the first several years before the debt-GDP ratio is stabilized, then, we need to calculate the final steady state repeatedly until $W_{G,T}$ is converged.

4 Calibration

The baseline economy is assumed to be in a steady-state equilibrium, thus on a balanced-growth path, with the current-law tax and social security system. Table 1 shows the main parameters values and baseline government policy values of the model economy. The discount factor, , is set so that the capital-output ratio, $K{=}Y$, to be 2.4; the depreciation rate of capital stock, , is chosen so that the real rate of return, r, is equal to 0.05; the growth-adjusted productivity, A, is set so that the wage rate, w, is normalized to unity in the baseline economy. For simplicity, the baseline government net worth, W_G, is assumed to be zero.

4.1 Demographics, Preference, and Technology Parameters

The maximum possible age, I, in the model economy is set to be $i = 80$, which corresponds to real age 100. The retirement age, I_R, is fixed at $i = 45$ (real age 65) for simplicity. The labor-augmenting productivity growth rate, , is 1.8% and the population growth rate, , is 1.0% in the model economy. The

[7] In Step 3, we obtain all of the decision rules and value functions in the transition path without updating a set of factor prices and government policy variables, $^0_1$. Thus, the procedure adopted here is Gauss-Jacobi iteration. Ríos-Rull (1999) explains a different solution algorithm that uses Gauss-Seidel iteration.

Table 1: Main Parameter Values and Baseline Government Policy Values

Maximum possible age	I	80	Real age 100
Retirement age	I_R	45	Full retirement age 65
Productivity growth rate		0.018	
Population growth rate		0.010	
Share parameter of consumption		0.36	
Discount factor		0.9873	K=Y $= 2{:}4$
Growth-adjusted discount factor	˜	0.9747	˜ $= \quad (1+ \quad)^{\alpha(1 \quad \gamma)}$
Coefficient of relative risk aversion		3.0	
Auto correlation parameter of log wage		0.95	
Standard deviation of log wage shocks		0.27	
Share parameter of capital stock		0.30	
Depreciation rate of capital stock		$0{:}075$	$r = 0{:}050$ in the baseline
Total factor productivity	A	0.9871	$w = 1{:}0$ in the baseline
Average median wage: men aged 21-65	$\bar{e}$	1.0	$w\overline{eh} \approx 0{:}36 = \$44{;}200$ in 2009 [1]
Income tax parameters: tax rate limit	$'_t$	0.30	Statutory rate $=$ 0.35 in 2009
: curvature	$'_1$	0.9601	o Estimated by OLS
: scale	$'_2$	1.0626	
: deduction/exemptions	d	0.1523	$2 \times \$3{;}650 + \$11{;}400$ in 2009
Social Security payroll tax rate	$^{-}{}_{P,t}$	0.10	OASI tax rate $\approx 0{:}106$=$1{:}053$
Maximum taxable earnings	$\#_{\max}$	0.8699	\$106,800 in 2009
OASI benefit adjustment factor	$_t$	1.3295	$TR_{SS,0} = T_{P,0}$
Replacement rate threshold: 0.90 & 0.32	$\#_1$	0.0727	\$744$\times 12 = \$8{;}928$ in 2009
: 0.32 & 0.15	$\#_2$	0.4382	\$4,483$\times 12 = \$53{;}796$ in 2009
Government consumption	$C_{G,0}$	3.1583	$C_{G,0} = T_{I,0} + T_{C,0}$
Lump-sum transfers	$tr_{LS,0}$	0.0169	$TR_{LS,0} = BQ_0$
Government net wealth	$W_{G,t}$	0.0	

*[1] The population average of the estimated median earnings of full-time male workers by age. A unit in the model economy thus corresponds to \$122,778 in 2009.

conditional survival rate at the end of each age, $_i$, is calculated from Table 4. C6 2005 Period Life Table in Social Security Administration (2010). The survival rate at the end of age 100 ($i = 80$) is replaced by zero.

The share parameter of consumption in the utility function, , is set at 0.36, following the real business cycle literature. The coefficient of relative risk aversion, , is set at 3.0. The share parameter of capital stock in the production function, , is set at 3.0. The depreciation rate of capital stock, , is 0.075 so that the interest rate, r, is equal to 0.05 in the baseline economy when the capital-output ratio 2.4. Total factor productivity, A, is 0.9871 so that the average wage rate, w, is normalized to unity in the baseline economy. The population-weighted average of the median wage rate, $w\bar{e}$, for ages 21-64 is also normalized to unity.

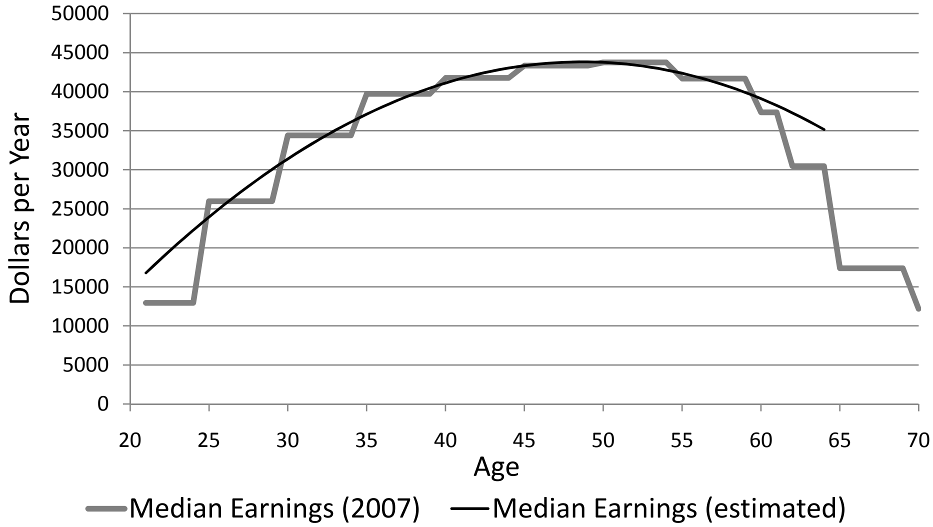

Figure 1: The Median Earnings of Male Workers by Age in 2007

4.2 Market Wage Processes

The working ability (hourly wage), e_i, of an age i household in the model economy is assumed to satisfy $\ln e_i = \ln \bar{e}_i + \ln z_i$ for $i = 1, \ldots, I_R - 1$, where $\bar{e}_i$ is the median wage rate at age i, and the persistent shock, z_i, follows an AR(1) process, $\ln z_i = \rho \ln z_{i-1} + \epsilon_i$, where $\epsilon_i \sim N(0, \sigma^2)$ and $\ln z_1 \sim N(0, \sigma^2_{\ln z_1})$. The median working ability, $\bar{e}_i$, for ages between 21 and 64 are constructed from Table 4.B6 Median Earnings of Workers by Age (male, 2007) in Social Security Administration (2010). The median earnings of all workers understates the working ability, because some workers choose not to work full time. The median earnings profile is extrapolated by using OLS and excluding ages 21-24 and 62-64 for possible schooling and early retirement. Figure 1 shows the original data and approximated values.

ture nodes, then 5 levels of $\ln z_i$ are generated by combining 4 nodes in each tail distribution into one node. The unconditional probability distribution of the 5 nodes is $_{j,i} = (0.0731, 0.2422, 0.3694, 0.2422, 0.0731)$ for $i = 1, \ldots, I_R - 1$. The Markov transition matrix of an age i household, $\Pi_i = [\ (e^{j'}_{i+1} \mid e^{j}_{i})]$, that corresponds to $= 0.95$ is calculated by using the bivariate normal distribution function as

$$\Pi_i = \begin{pmatrix} 0.8979 & 0.1021 & 0.0000 & 0.0000 & 0.0000 \\ 0.0308 & 0.8902 & 0.0790 & 0.0000 & 0.0000 \\ 0.0000 & 0.0518 & 0.8964 & 0.0518 & 0.0000 \\ 0.0000 & 0.0000 & 0.0790 & 0.8902 & 0.0308 \\ 0.0000 & 0.0000 & 0.0000 & 0.1021 & 0.8979 \end{pmatrix}$$

for $i = 1, \ldots, I_R - 1$.

4.3 Government's Policy Functions

The individual income tax function is one estimated in Gouveia and Strauss (1994). The parameters of the progressive tax function are estimated by OLS with the statutory marginal tax rates in 2009. One of the parameters, $'_t$, is the limit of the marginal tax rate as taxable income goes to infinity. First, $'_t$ is set at 0.35, the highest marginal tax rate in 2009, and the other two parameters, $'_{m,1}$ and $'_{m,2}$, are estimated by OLS (equally weighted for taxable income between \$0 and \$500,000), assuming households are married. Then, $'_t$ is reduced to 0.30 from 0.35 to reflect the lower effective income tax rates. With this lowered parameter, individual income tax revenue, $T_{I,t}$, in the baseline economy is 9.6% of GDP, which is consistent with the U.S. economy. Figure 2 shows the statutory, approximated, and effective marginal income tax rates.

The OASI payroll tax rate is 5.3% for an employee and 5.3% for an employer. The payroll tax rate, $^-_{P,t}$, for earnings below the maximum taxable earnings is set at 0.10, which is approximately equal to $10.6/105.3$. In the current U.S. social security system, the thresholds to calculate primary insurance amounts (PIA) are set for each age cohort when a worker reaches age 62. In the model economy, the growth-adjusted thresholds are fixed for all age cohorts, and the PIA is adjusted later by using the long-term productivity growth rate and years from age 60. Thus, the model simply uses the thresholds for the age 62 cohort in 2009 after scale adjustment. The OASDI benefit adjustment factor, $_t$, is set at 1.3295 so that the OASI budget is balanced in the baseline economy. The additional 0.3295 of benefits are roughly consistent with the spousal and survivors benefits.

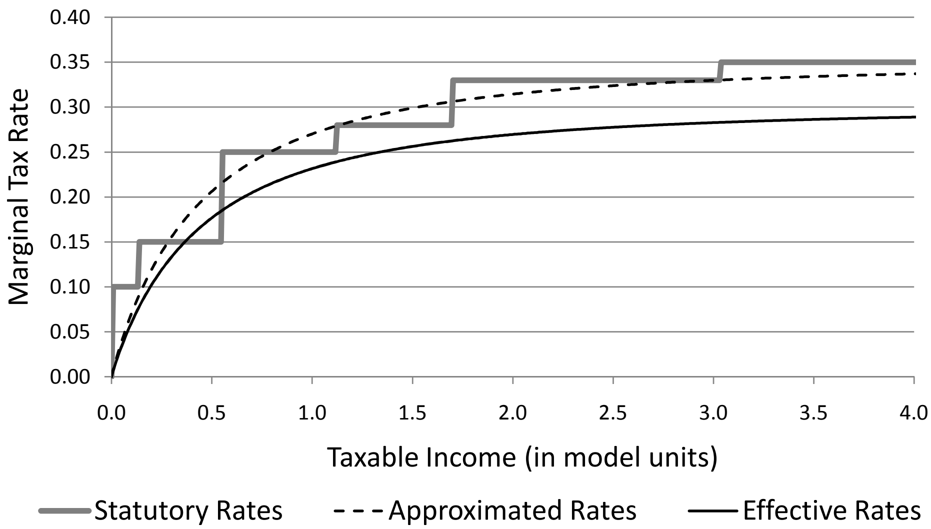

Figure 2: The Marginal Income Tax Rate Schedule of Married Households

5 Policy Experiments

This section explains two examples of policy experiments. In both cases, the economy is assumed to be in a steady-state equilibrium, or equivalently, on the balanced growth path in year 0. At the beginning of year 1, the government announces and introduces a permanent (possibly phased-in) fiscal policy change. The government policy is assumed to be credible in the model economy. After the policy change, the economy is approaching to a new steady-state equilibrium.

In the first experiment, the government cuts the marginal income tax rates proportionally by 50% and finances the revenue reduction by increasing the consumption tax rate so that the government budget is balanced in each year throughout the transition path. This experiment is similar to that in Nishiyama and Smetters (2005). In the second experiment, the government introduces a "partial privatization" of the social security pension. The government cuts the OASI benefits by 50% in a phased-in manner, cohort by cohort, for the first 40 years and reduces the payroll tax rate to balance the social security budget in each year. The second experiment is similar to that in Nishiyama and Smetters (2007).

5.1 Consumption Tax Reform

In this section, we assume that the government reduces the marginal income tax rates proportionally by 50% at the beginning of period 1 and keep the tax rates at the same levels over time. Then, the government increases the consumption tax rate to balance the non-social-security budget period by period. Because the

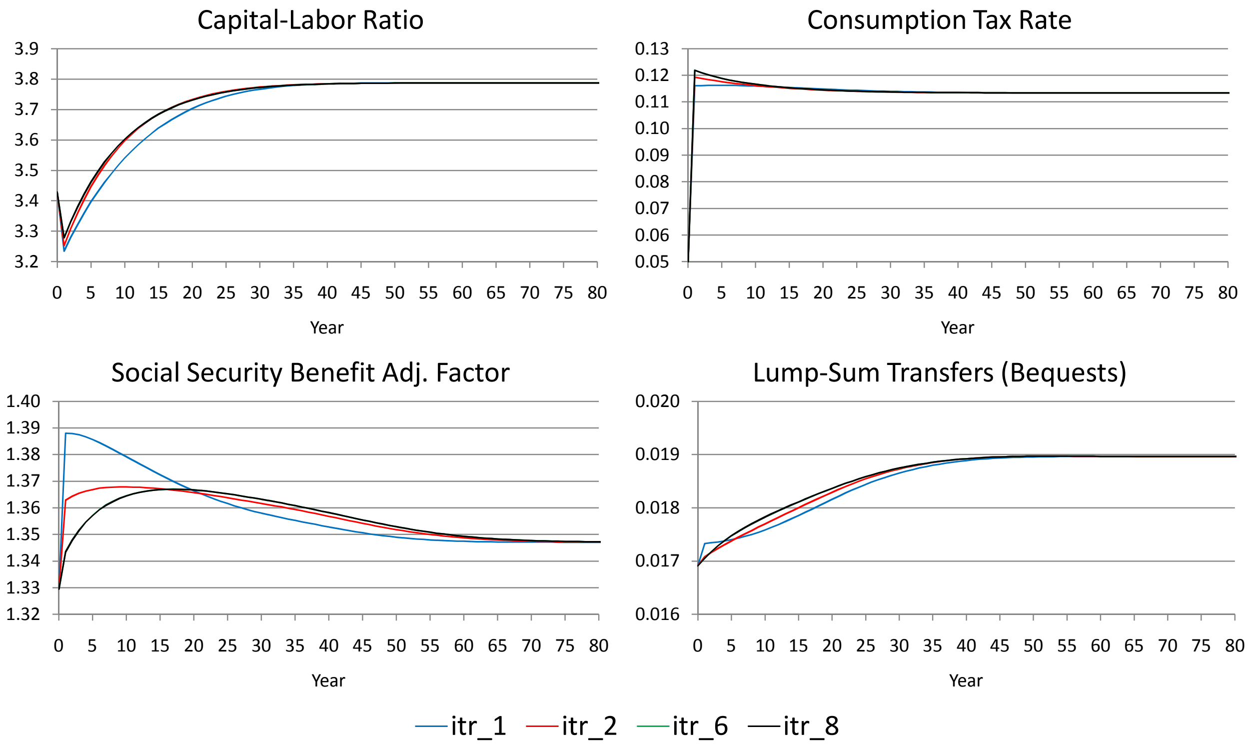

Figure 3: The Iteration Process of Consumption Tax Reform

policy change alters the size of payroll tax revenue through higher labor income, the government also adjusts pay-as-you-go benefits proportionally to balance the social security budget over time.

More specifically, the government reduces the individual income tax rate parameter, φ_t, to 0.15 from 0.30 for periods $t = 1, \ldots, \infty$. Then, it increases the consumption tax rate, $\tau_{C,t}$, and adjusts the OASI benefit adjustment factor, λ_t, in each period to balance the government budget, i.e., for all t, $W_{G,t} = W_{G,0}$, and

$$T_{C,t}(\tau_{C,t}) = C_{G,t} + TR_{LS,t}(tr_{LS,t}) - T_{I,t}(\varphi_t) - BQ_t + (1+\mu)(1+\nu)W_{G,t+1} - (1+r_t)W_{G,t};$$

$$TR_{SS,t}(\lambda_t) = T_{P,t}(\bar{\tau}_{P,t}).$$

Figure 3 shows the iteration process of the capital-labor ratio and the endogenous government policy variables. The model is solved for $T = 150$ periods, and the economy in period T is assumed to be in the final steady-state equilibrium. The initial guesses of these variables in $t = 1, \ldots, 149$ are set to be equal to those in the final steady state. In this simple policy experiment, it takes only 8 iterations to reach

Table 2: The Effects of Consumption Tax Reform (changes from the baseline economy)

Year	1	10	20	50	Long run
Capital Stock (National Wealth)	0.0	9.1	12.8	14.4	14.5
Labor Supply (in Efficiency Units)	4.6	3.8	3.6	3.6	3.6
Gross Domestic Product	3.2	5.4	6.3	6.8	6.8
Welfare of Age 21 Households	-2.7	-1.5	-1.0	-0.7	-0.7
Income Tax Rates	-15.0	-15.0	-15.0	-15.0	-15.0
Consumption Tax Rate	7.2	6.7	6.5	6.3	6.3

Macroeconomic variables are % changes from the baseline economy. Tax rates are changes in percentage points from the baseline economy.

the equilibrium transition path.[8] Although the capital-labor ratio and the endogenous policy variables in iterations 1 and 2 are noticeably different from those in iteration 8, the values in iterations 6 and 8 are not distinguishable from the graph.

Table 2 summarizes the effects of the policy change. After the marginal income tax rates are cut in half, the consumption tax must be increased by 7.2 percentage points in the first year and, after the economy grows, by 6.3 points in the long run. The labor supply increases (in efficiency units) by 4.6% in the first year and 3.6% in the long run. The capital stock increases by 9.1% in year 10 and 14.5% in the long run. Accordingly, gross domestic product increases by 3.2% in the first year and 6.8% in the long run.

Although the changes in the macroeconomic variables are all positive throughout the transition path, the average welfare of newborn (age 21) households will be deteriorated both in the short run and in the long run. The age 21 households in the first year are worse off by 2.7% using the consumption-equivalence measure described earlier. Yet, the age-21 households that are born in the long run will also be worse off by 0.7%. Younger households, both in the short run and long run, tend to benefit from the implicit "lump-sum levy" on existing lifecycle wealth that was accumulated by older households at the time of reform. They also benefit from the reduced distortions to savings, which lowers the relative price of future consumption. However, there are two competing economic effects: the consumption base is smaller than the income tax base; it is also less progressive. The smaller size of the tax base requires larger distorting tax rates on labor supply; the reduction in progressivity removes some of the tacit insurance against negative wage shocks which are otherwise uninsurable. Indeed, increasing the coefficient of relative risk aversion tends to also

[8]When we assume a temporary deficit-financing government policy change, however, it will take more iterations to find the equilibrium transition path. This is because capital-labor ratio is directly affected by the increase in the government debt, and the government debt is sensitive to the interest rate.

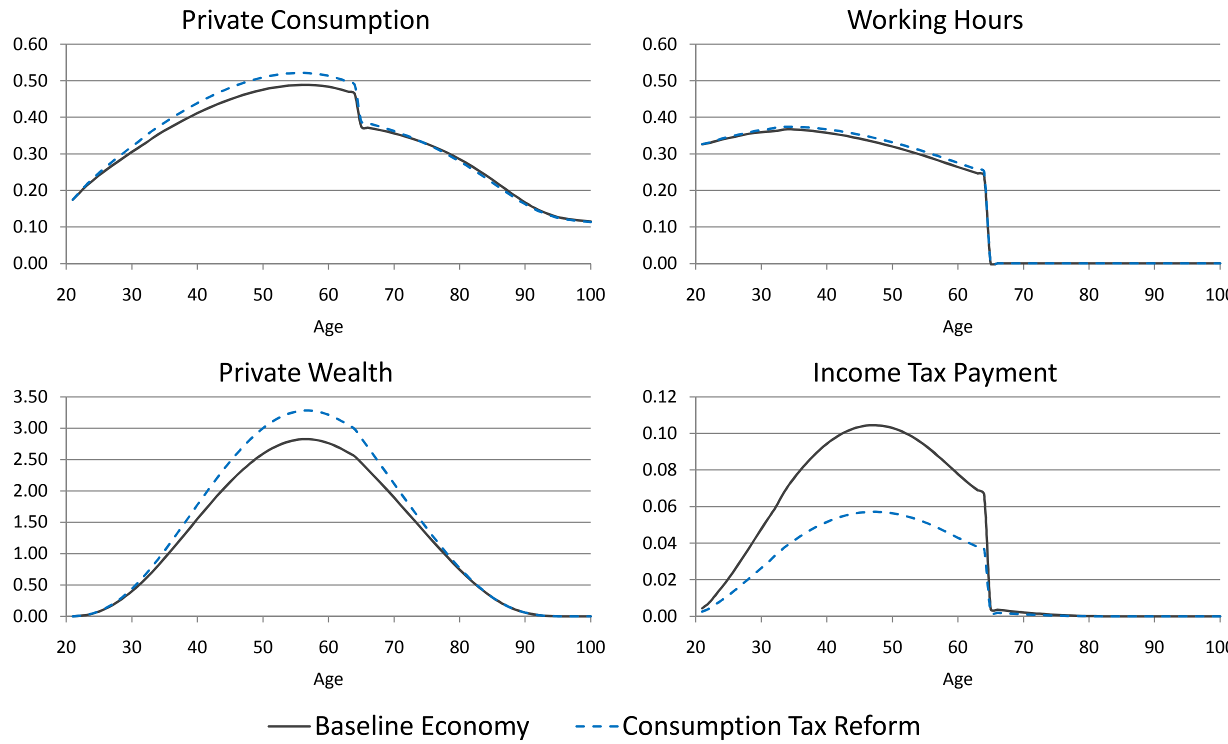

Figure 4: The Long-Run Effects of Consumption Tax Reform

increase the welfare loss.

Figure 4 shows the long-run effects of the consumption tax reform over the life cycle. Working hours increase slightly, especially during the middle age of life when earnings are the largest. Private consumption will increase significantly before the retirement due to larger wages and a lower interest rate. To prepare for the larger consumption tax payment after the retirement, households accumulate larger lifecycle savings.

Figure 5 shows the effects of the consumption tax reform throughout the transition path. Because the policy change is relatively simple, the economy converges almost entirely to its long-run outcome within 50 years after the policy change. In the short run, the implicit "lump-sum levy" hurts older households who are alive at the time of the change. This negative wealth effect causes labor supply to increase more in the short run than in the long run, and the welfare loss of current households will be much larger than that of the future households.

5.2 Social Security Privatization

Now consider a different policy reform where the government reduces the OASI benefits by 50%, phased in linearly over the next 40 years. Specifically, for households aged 61 ($i = 41$) or older in period 1, their

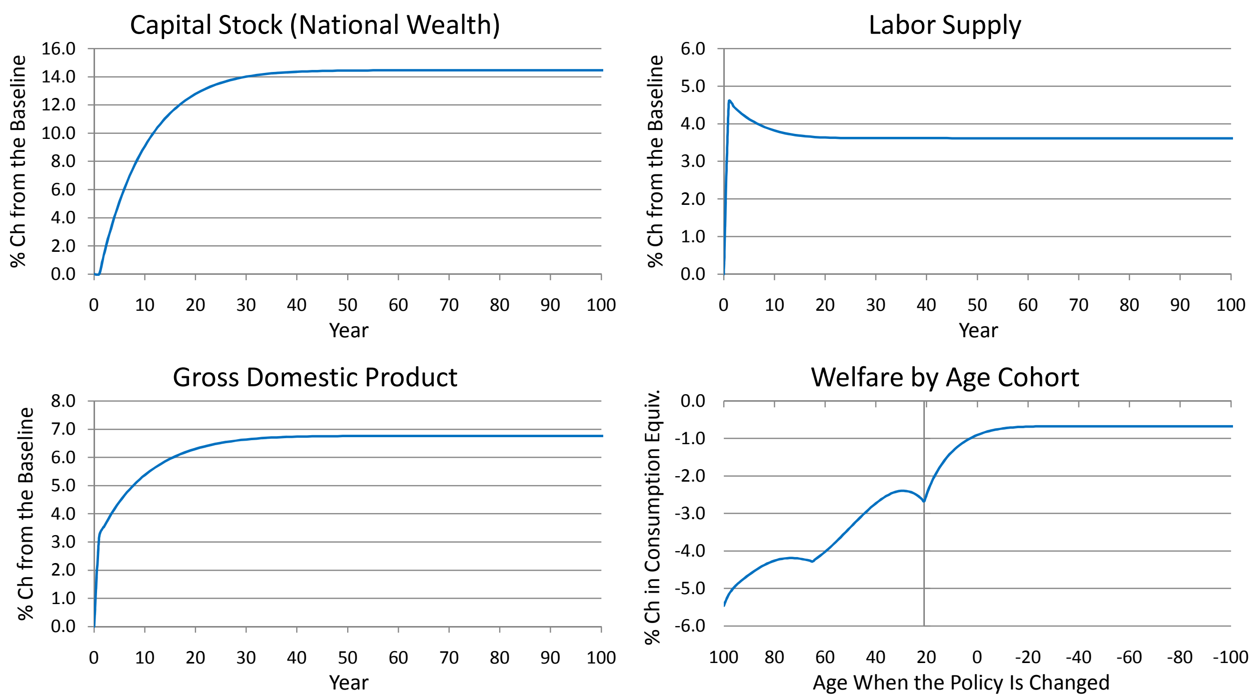

Figure 5: The Transition Effects of Consumption Tax Reform

benefit adjustment factor is unchanged, i.e.,

$$ {}^{i}_{t} = {}_{0} \quad \text{for} \quad i - (t-1) \geq 41; \quad t \geq 1. $$

For households aged 21 ($i = 1$) or younger in period 1, their benefit adjustment factor is reduced by 50%, i.e.,

$$ {}^{i}_{t} = {}_{T} = 0.5 \, {}_{0} \quad \text{for} \quad i - (t-1) \leq 1; \quad t \geq 1. $$

Finally, for households aged between 22 ($i = 2$) and 60 ($i = 40$), their benefit adjustment factor is the weighted average of the above two parameter values, i.e.,

$$ {}^{i}_{t} = \frac{i-t}{40} \, {}_{0} + \left(1 - \frac{i-t}{40}\right) {}_{T} \quad \text{for} \quad 2 \leq i - (t-1) \leq 40; \quad t \geq 1. $$

Then, under the balanced-budget assumption, $W_{G,t} = W_{G,0}$ for all t, the government reduces the Social Security payroll tax rate, $\bar{}_{P,t}$, gradually to balance the social security budget in each period. It also changes the individual income tax rates proportionally by adjusting $'_{t}$ to balance the rest of the government budget

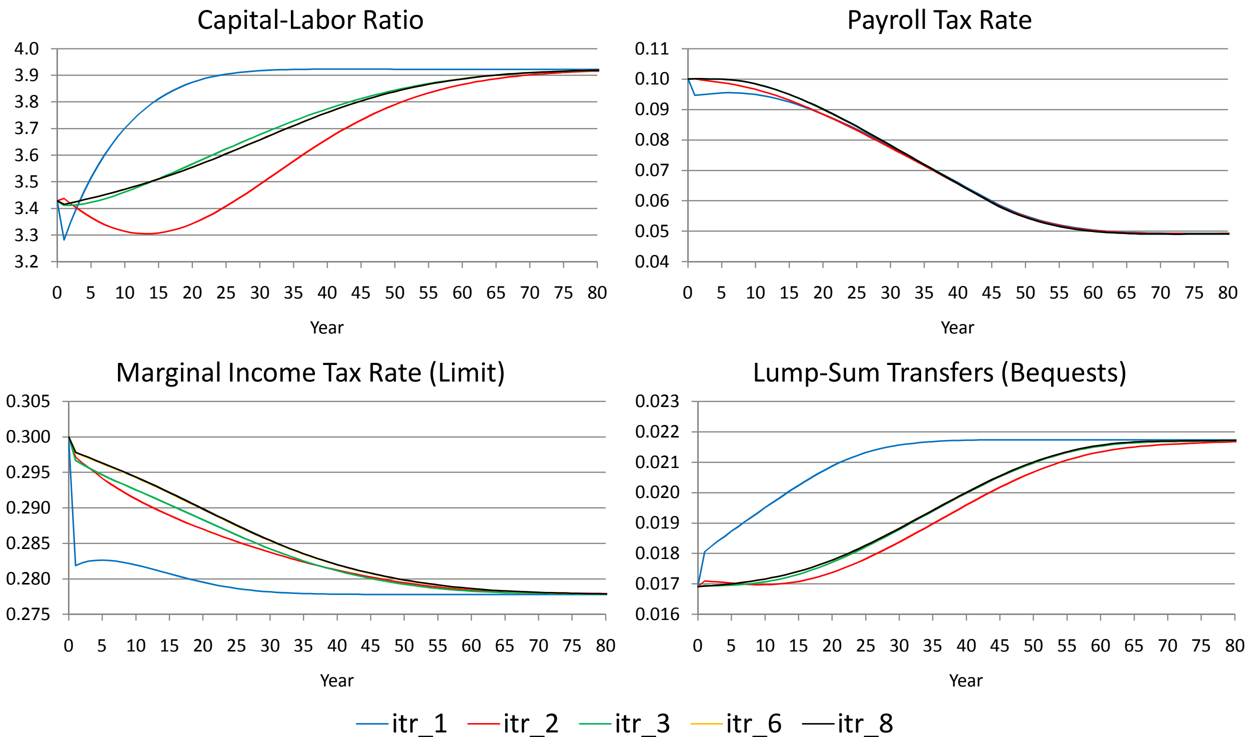

Figure 6: The Iteration Process of Consumption Tax Reform

period by period, i.e., for all t, $W_{G,t} = W_{G,0}$, and

$$T_{P,t}(\bar{\ }_{P,t}) = TR_{SS,t}(\{\ {}^{i}_{t}\}_{i=1}^{I});$$

$$T_{I,t}(\ '_{t}) = C_{G,t} + TR_{LS,t}(tr_{LS,t}) - T_{C,t}(\ _{C,t}) + (1+\ \ )(1+\ \ )W_{G,t+1} - (1+r_t)W_{G,t}:$$

Figure 6 shows the iteration process of the capital-labor ratio and the endogenous government policy variables. Similar to the previous policy experiment, the model economy reaches to the equilibrium transition path in 8 iterations. Although the capital-labor ratio and the endogenous policy variables change significantly for the first 3 iterations, the values in iterations 6 and 8 are, again, not distinguishable from the graph. The error in the first iteration is large because the initial guess of the payroll tax rate does not take the phased-in policy change in to account. A better initial guess can possibly reduce the iteration count by one.

Table 3 shows the effects on aggregate and policy variables. Lifecycle saving for retirement obviously increases, raising the capital stock by 2.0% in year 10, 14.6% in year 50, and 17.0% in the long run. Gross domestic output will also increase by 1.1% in year 10 and 6.5% in the long run. The payroll tax, however,

Table 3: The Effects of Social Security Privatization (changes from the baseline economy)

Year	1	10	20	50	Long run
Capital Stock (National Wealth)	0.0	2.0	5.0	14.6	17.0
Labor Supply (in Efficiency Units)	0.4	0.7	1.2	2.3	2.3
Gross Domestic Product	0.3	1.1	2.3	5.9	6.5
Welfare of Age 21 Households	-1.6	-0.2	1.8	6.3	6.8
Income Tax Rates	-0.2	-0.6	-1.0	-2.0	-2.2
Payroll Tax Rate	0.0	-0.2	-1.0	-4.5	-5.1

Macroeconomic variables are % changes from the baseline economy. Tax rates are changes in percentage points from the baseline economy.

falls only gradually over time because (1) the benefit cuts are phased in over 40 years and (2) the payroll tax base grows over time as the capital stock and wages increase. The payroll tax rate, therefore, is almost unchanged in the first year but falls by almost 5.1 percentage points in the long run. Labor supply increases by 0.4% in the first year and 2.3% in the long run.

Similar to the consumption tax reform experiment, the changes in the macroeconomic variables are all positive throughout the transition path. However, the welfare results are very different. Age-21 households alive at the time of the reform are hurt by (1.6% in consumption equivalence) because they help pay for the policy transition path during their working years but collect a substantially reduced benefit upon retirement. [9] However, age-21 households born in the long run are better off (by 6.8%) because they are born into a world with a smaller social security system. While they receive a smaller social security benefit, they also pay less into the steady state system, which they prefer when the economy is dynamically efficient (i.e., the interest rate exceeds the growth rate of the payroll tax base).

Figure 7 shows the long-run effects of the social security privatization over the lifecycle. Working hours decrease slightly when households are young but increase in their middle ages. Private consumption increases before the retirement and decreases after the retirement because the wage rate is higher and the interest rate is lower. Households accumulate larger lifecycle wealth after the policy change in order to replace the reduction in social security benefits.

Figure 8 shows the effects of social security privatization during the transition path. Because the policy experiment is phased in over 40 years, it takes much longer (almost 100 years) to reach the new steady-state

[9] The welfare loss in the short run is partially exacerbated by the reduced redistribution from the OASI system. The short run welfare loss is smaller if the income tax rates are instead changed in a *progressive* manner in order to finance the transition cost.

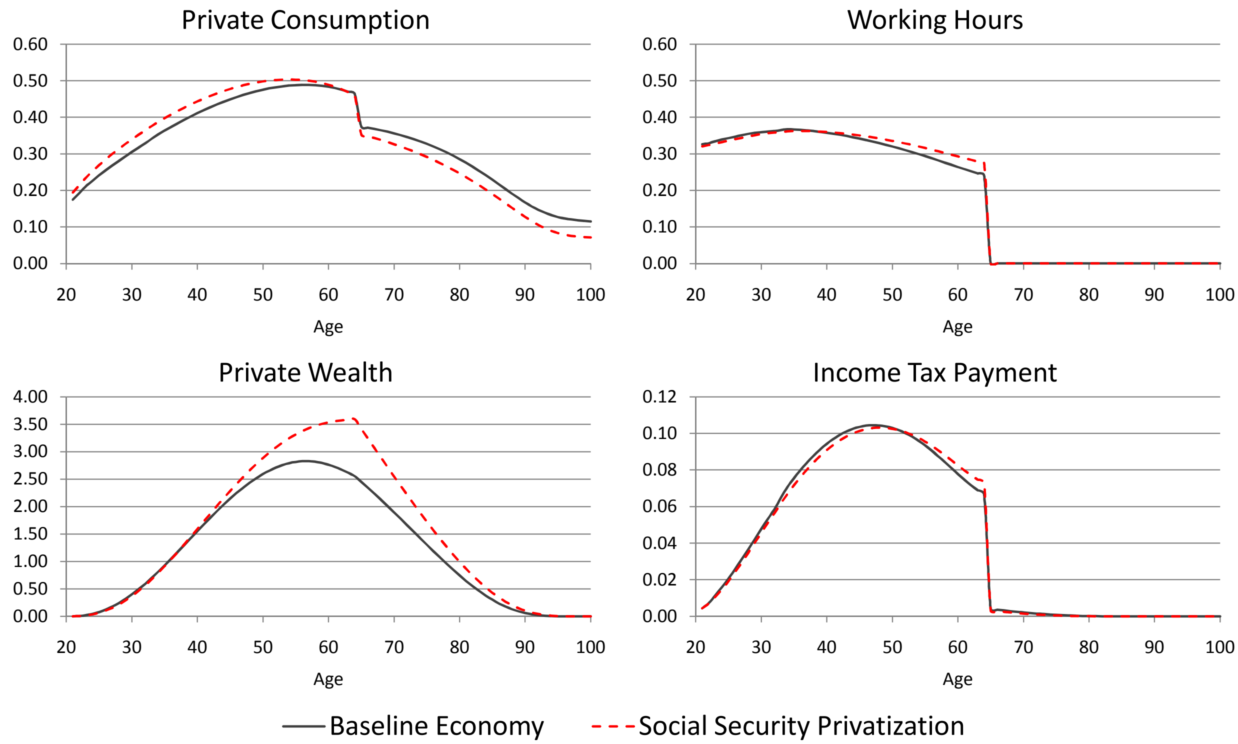

Figure 7: The Long-Run Effects of Social Security Privatization

economy relative the tax policy experiment considered above. Notice that the impact on household welfare is very non-monotonic in their age at the time of the reform. Households aged 61 or older at the time of the policy change are protected, by construction, from any reduction in benefits. However, they are slight better off due to the small reduction in income tax rates. However, households aged between 12 and 60 at the time of the reform are worse off, because they help pay for the policy transition path but receive lower future benefits when they retire. Households born in the long run are obviously better off.

6 Concluding Remarks

The heterogeneous-agent overlapping-generations framework is the main workhorse for analyzing fiscal policy changes with possible *intra*-generational and *inter*-generational wealth redistribution. The pioneering early work of Auerbach and Kotlikoff (1987) showed how to solve a deterministic OLG model with limited heterogeneity by using standard Euler-equation methods. However, the OLG model is computationally challenging to solve in the presence of the wide range of heterogeneity and uncertainty. This chapter extends their work by showing how to solve the heterogeneous-agent OLG model with idiosyncratic uncertainty by

Figure 8: The Transition Effects of Social Security Privatization

deploying recursive methods. Adding idiosyncratic uncertainty allows us to recognize the impact that policy changes might have on the pooling of idiosyncratic risks which are typically hard to insure in the private market. This specific channel often has first-order effects.

The computational algorithms explained in this chapter are quite straightforward. We discretize the state space of a heterogeneous household. For each node and each period, we solve the household problem by using a standard Newton-type nonlinear equation solver and a linear or quadratic interpolation. Using the household's decision rules, then, we obtain the time series of factor prices and government policy variables (parameters) by Gauss-Jacobi type iterations.

Although there are not any surprising tricks in the procedure, the computation is fairly efficient. When the number of ages is $I = 80$ (e.g., $i = 21, \ldots, 100$), the number of asset nodes is $J = 61$, the number of average historical earnings nodes is $K = 20$, and the number of working ability nodes is $L = 5$, for example, it only takes 3 seconds per period per iteration in a PC with Intel Core i7-930 (2.8GHz) processor. Because most policy experiments take only 8 to 10 iterations of factor prices and policy variables to converge, we can solve the model for steady-state equilibrium within 30 seconds and for an equilibrium transition path of 150 periods within a couple of hours.

Thus, it will not cause a serious problem to add a couple of decision variables. Since we use a Newton-type iteration to solve an individual problem, the computational cost will increase only quadratically. However, the computation will be much more costly if we add a state variable. If the new state variable is descritized into 20 nodes, it will in general take 20 times longer to solve the model and will take much more memory space. We need to consider using a better interpolation method to reduce the total number of nodes in the state space.[10]

The much bigger challenge, however, is adding *aggregate* uncertainty to the heterogeneous-agent OLG framework herein, thereby allowing for stochastic factor prices and policy variables. Unfortunately, the "curse of dimensionality" quickly takes over since the distribution of heterogeneous households must be indexed across many aggregate states. While various perturbation and bifurcation techniques might someday be recruited to help solve models with aggregate uncertainty, that frontier still appears to be very far away.

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