Two-dimensional screening: a useful optimality condition☆

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Abstract

This paper studies single-product and two-dimensions of heterogeneity monopolistic screening problems. It makes a connection between one-dimensional screening without single-crossing and two-dimensional screening. We show that the optimality condition characterizing discrete bunching of types that emerges in one-dimensional models can be extended to two-dimensional models. Then, we use this condition to solve some examples from the literature.

Keywords: two-dimensional screening, bunching, non single-crossing

JEL: D42, D82.

1. Introduction

We study a monopolistic screening problem where the firm produces a single product and faces heterogeneous customers. The customers' preferences exhibit two dimensions of uncertainty, which are their private information. In this paper we will derive the necessary optimality conditions characterizing the level curves for the optimal quantity assignment function.

Due to dimensionality constraints, we do not expect to observe a complete separation of customers with regard to the quantity q of the good purchased. Indeed, it is natural to observe a bunching of customers attached to each quantity q sold by the firm. Consider the quantity assignment function $q(\cdot)$. The level curve $q(\cdot) = q$ represents exactly the bunching of customers at this level of consumption.

The approach we use is based on a natural change of variables in the customers type's space. The new variables will be related to the level curves of the quantity assignment function $q(\cdot)$. To find the change of variables, we use the method of characteristic curves from the theory of quasilinear partial differential equations. Once we have them, we proceed with a calculus of variations using

☆ *Revision 1.226–June 4, 2012*
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these new variables. This will result in an optimality condition characterizing the optimal level curves of the quantity assignment function.

In the one-dimensional type case without single-crossing, Araujo and Moreira [1] performed an analysis similar to ours, deriving a condition characterizing the optimal bunching. However, in their case, the bunching consists of only two customers choosing the same quantity q. On the other hand, we deal with a more complex structure as we have a continuum of customers choosing q. In this sense, this paper makes a connection between one-dimensional screening without single-crossing and two-dimensional screening. Moreover, we can think of the optimality conditions derived in this paper as extensions to the two-dimensional context of the analogous conditions derived by Araujo and Moreira [1] in the one-dimensional context.

Finally, with the conditions derived here, we can easily solve the examples from Laffont et al. [10] and the very interesting variation proposed by Deneckere and Severinov [6], where the level curves of $q(\cdot)$ may interact with the boundary curve separating the participation and the exclusion region on the type space.

1.1. Related Literature

In the discussion that follows, M is the number of instruments available to the monopolist and N is the dimension of the customer's heterogeneity (i.e. the dimension of the type space).

We observe that the solution of the monopolist's problem is straightforward in the one-dimensional case ($M = N = 1$) under the single-crossing condition. The problem translates into a well-behaved maximization problem on the space of monotonic functions.[1] However, in the multi-dimensional case ($M > 1$ or $N > 1$), the characterization of the optimal contract is a very complex problem in general.

Laffont et al. [10] presented a particular nonlinear pricing example where the monopolist sells a single product and is uncertain about the intercept and the slope of the customer's demand curve, i.e. ($M = 1, N = 2$). They derived the optimal quantity assignment function when these two characteristics are independently and uniformly distributed.

McAfee and McMillan [11] studied the problem of a multi-product monopolist who faces a customer with multi-dimensional characteristics with ($M \leq N$). They introduced the Generalized Single-Crossing (GSC), a condition under which the first- and second-order conditions for the customer's problem are necessary and sufficient for implementability. They also characterized the optimal contract when ($M = 1, N \geq 1$) under the (GSC), generalizing the result from Laffont et al. [10].

Armstrong [2] considered a problem where ($M \geq 1, N \geq 1$). He was able to give closed form solutions for some examples in this multidimensional context. One of his main contributions was to discover the exclusion property. This

[1] See Fudenberg and Tirole [7], chapter 7.

property states that the optimal contract leaves a set of customers, with positive measure, excluded from consumption.

Rochet and Chone [15] established the existence of the optimal contract and provided the characterization in the case of a multi-product monopolist who faces a customer with multidimensional characteristics. They assume the number of products and characteristics to be the same ($M = N \geq 1$) and also that the parametrization of a customer's preferences is linear with types. They introduced the sweeping procedure as a generalization of the ironing procedure for dealing with bunching in the multidimensional context.

Basov [3] introduced the Hamiltonian approach as a tentative method of generalizing Rochet and Chone [15] to the case when the number of products and characteristics may be different ($N \geq 1, M \geq 1$). Later on, these techniques were extended in his book (Basov [4]) to deal with more general customer preferences.

Parallel to this literature, there were works analyzing the existence of a solution for the monopolist's maximization problem. We highlight the papers by Monteiro and Page Jr [13] and by Carlier [5]. The former uses compactness properties resulting from budget constraint considerations and the latter uses direct methods and concepts of abstract convexity. However, these approaches do not actually provide any recipe for the characterization of the optimal contract.

We can say that Laffont et al. [10] is the main inspiration for our paper. We develop further some ideas that were already in their example, but now apply them in a more general context.

1.2. Outline of the Paper

The plan of the paper is the following. In Section 2 we present the model used in our analysis. In Section 3 we derive the partial differential equation related to the incentive compatibility constraints. The method for solving this equation gives us a reparametrization of the type space. In Section 4 we use this reparametrization to derive the optimality conditions. In Section 5 we use our method to solve two examples from the related literature. Finally, in Section 6 we give the conclusions. All proofs are relegated to the Appendix.

2. Model

The customer has a quasi-linear preference, represented by

$$v(q, a, b) - t,$$

where $(a, b) \in \Theta = [0, 1] \times [0, 1]$ is the customer's type, $q \in \mathbb{R}_+$ is the good consumed, and t is the monetary payment.

The firm is a profit maximizing monopolist which produces a single product $q \in \mathbb{R}_+$. The firm does not observe (a, b) and has a prior distribution over Θ according to the differentiable density function $f(a, b) > 0$. The monopolist's preference is given by

$$\Pi(q, t) = t - C(q),$$

where $C(q)$ is a C^2 cost function, with $C(\cdot) \geq 0, C'(\cdot) > 0$, and $C''(\cdot) > 0$.

Using the *'Revelation Principle'* we can restrict our attention to direct and truthful mechanisms.[2] Thus, the monopolist's problem consists in choosing the contract $(q, t) : \Theta \to \mathbb{R}_+ \times \mathbb{R}$ that solves

$$\max_{q(\cdot),t(\cdot)} \int_0^1 \int_0^1 \Pi(q(a, b), t(a, b)) f(a, b) da db, \tag{Π}$$

subject to the *individual-rationality* constraints:

$$v(q(a, b), a, b) - t(a, b) \geq 0 \quad \forall (a, b) \in \Theta, \tag{IR}$$

and the *incentive compatibility* constraints:

$$(a, b) \in \arg \max_{(a',b')\in\Theta} \{v(q(a', b'), a, b) - t(a', b')\}, \quad \forall (a, b) \in \Theta. \tag{IC}$$

A contract $(q(\cdot), t(\cdot))$ is incentive-compatible if it satisfies the (IC) constraints. We say that $q(\cdot)$ is implementable if we can find a monetary payment $t(\cdot)$ such that the pair $(q(\cdot), t(\cdot))$ is incentive compatible. For an incentive-compatible contract, we define the informational rent as

$$V(a, b) = v(q(a, b), a, b) - t(a, b). \tag{1}$$

In the single-product, single-characteristic case ($M = N = 1$), the informational rent is used to eliminate the monetary payment $t(\cdot)$ from the monopolist's problem (Π). After that, combining integration by parts and the envelope's theorem from Milgrom and Segal [12] one can derive a new expression for the monopolist's expected profit now depending only on $q(\cdot)$ and the customer's type.[3]

This idea can be extended to the multi-dimensional context. Indeed, in the case with multiple characteristics ($N > 1$), Armstrong [2] proposed the "integration by rays" technique, that also results in an expression for the monopolist's

[2] The *'Revelation Principle'* has been enunciated in Gibbard [8].
[3] See Fudenberg and Tirole [7], chapter 7, for more details.

expected profit depending only on $q(\cdot)$ and the customer's type. However, when we have multiple characteristics, there are several paths connecting distinct customers. So, instead of using "integration by rays", it may be more convenient to choose a different path for this integration. This decision, of course, depends on the specific problem to be addressed.

In this paper, we are not concerned with the particular method used. We will assume that the monopolist's expected payoff is given by

$$\int_0^1 \int_0^1 g(q(a,b),a,b)f(a,b)dadb. \tag{2}$$

As in the one-dimensional case, we will call $g(\cdot)$ the virtual surplus. To simplify our notation, we define $G(q,a,b) = g(q,a,b)f(a,b)$. We make the following assumptions about the utility function $v(q,a,b)$ and the function $G(q,a,b)$.

Assumption A. *$v(q,a,b)$ is thrice differentiable and $G(q,a,b)$ is twice differentiable. They satisfy:*

A1. *$v_a > 0$ and $v_b < 0$ when $q > 0$.*

A2. *$v_{qa} > 0$ and $v_{qb} < 0$ when $q > 0$.*

A3. *$v_{q^2} < 0$ and $G_{q^2} < 0$.*

Observe that assumption **A**1 implies that the informational rent increases with a and decreases with b. Assumption **A**2 is a single-crossing condition in each direction a and b. As a consequence, it requires that an implementable $q(a,b)$ is increasing with a and decreasing with b. Finally, assumption **A**3 requires the strict concavity of the utility function $v(\cdot,a,b)$, and the strict concavity of $G(\cdot,a,b)$, for each (a,b)-type customer. The last assumption assures that the first-order necessary conditions for a $q(a,b)$ that maximizes expression (2) are also sufficient.

3. Local Incentive Conditions

Now we present the partial differential equation (PDE) that is derived from the (IC) constraints. First, consider an incentive compatible contract (q,t). Then, each (a,b)-type customer must solve the maximization problem

$$\max_{(a',b')\in\Theta} \{v(q(a',b'),a,b) - t(a',b')\}. \tag{3}$$

The first-order necessary optimality conditions for problem (3) are

$$\begin{cases} v_q(q(a,b),a,b)q_a(a,b) = t_a(a,b), \text{ and} & (4)\\ v_q(q(a,b),a,b)q_b(a,b) = t_b(a,b). & (5)\end{cases}$$

From equations (4) and (5), we can derive the cross derivatives t_{ab} and t_{ba}. Finally, using the Schwarz's integrability condition

$$t_{ab}(a,b) = t_{ba}(a,b),$$

we get Proposition 1 below.[4]

Proposition 1 (Quasi-Linear Equation)**.** *Suppose that the contract* (q, t) *is incentive compatible and twice differentiable in an open set* $\Omega \subset \Theta$*. Then it satisfies the following equation*

$$-\frac{v_{qb}}{v_{qa}} q_a + q_b = 0. \tag{6}$$

Equation (6) is a quasi-linear first-order partial differential equation.[5] It describes the relationship between the contour lines of $q(a, b)$ and $v_q(q(a, b), a, b)$. Indeed, let $(a(s), b(s))$ be a contour line with $q(a(s), b(s)) = k$. Then, differentiating $q(a, b)$ and $v_q(q(a, b), a, b)$ along this curve, we get

$$\frac{d}{ds} q(a(s), b(s)) = q_a a_s + q_b b_s = 0, \tag{7}$$

and

$$\frac{d}{ds} v_q(k, a(s), b(s)) = v_{qa} a_s + v_{qb} b_s. \tag{8}$$

Finally, using equations (6),(7), and (8), we conclude that

$$\frac{d}{ds} v_q(k, a(s), b(s)) = 0. \tag{9}$$

Observe that equation (9) is saying that if $(a(s), b(s))$ is a contour line $q(a, b) = k$, then it is also a contour line of $v_q(k, a, b)$.

We have yet another interpretation. The "Taxation Principle" says that we can also implement $q(a, b)$ with a tariff $P : Q = q(\Theta) \to \mathbb{R}$, $P(q(a, b)) = t(a, b)$ for all $(a, b) \in \Theta$. Using this tariff P, we can write the customer's problem as

$$\max_{q \in Q} v(q, a, b) - P(q).$$

Notice that, when P is differentiable at $q(a(s), b(s)) = k$, Proposition 1 is simply saying that all the types choosing $q(a(s), b(s))$ get the same marginal utility $v_q(k, a(s), b(s))$. In Araujo and Moreira [1], they have a similar condition (the U-condition) in the one-dimensional context. They use it in the derivation of the optimality condition. We will follow the same steps, adapted to the two-dimensional case.

[4] Laffont et al. [10] mentioned this integrability condition and the PDE for the particular case they were treating. Here we present the general expression for this PDE.

[5] See John [9] for a complete analysis of this kind of PDE and a description of the method of characteristic curves used to solve it.

3.1. Solving the Quasi-Linear Equation

The solution of the quasi-linear equation (6) will provide a natural reparametrization of types in the participation region. This reparametrization will follow the contour lines of $q(a,b)$. After that, using calculus of variations, we will derive an optimality condition involving types in the same contour line $q(a,b)=k$.

We use the method of characteristic curves to solve equation (6). This method consists in reducing a partial differential equation to a system of ordinary differential equations. Then, the system is integrated using the initial data prescribed on a curve Γ.[6] Formally, we have the following Cauchy initial value problem associated with equation (6):

$$\begin{cases} -\dfrac{v_{qb}}{v_{qa}} q_a + q_b = 0, \\ \\ q_{|\Gamma} = \phi(r), \end{cases} \tag{CP}$$

where $\Gamma = \{(\alpha_0(r), \beta_0(r))\}$ is a curve on the ab-plane, defined on $I = [\underline{r}, \overline{r}]$. In this curve, we have $q(\alpha_0(r), \beta_0(r)) = \phi(r)$. The basic idea is to prescribe the value of $q(\cdot)$ on Γ and then use the characteristic curves to propagate this information to the participation region, as we can see in Fig. 1. In this sense, because Γ is a one-dimensional curve, we are reducing problem from two-dimensions to one.

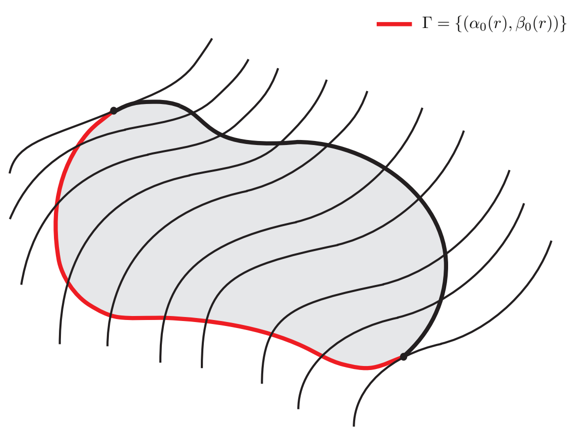

Figure 1: The characteristic plane curves

[6] For a description of the method, we refer the reader to John [9] and Petrovski [14].

Following the method, we define the family of curves $(a(r,s), b(r,s), z(r,s))$ as the solution of

$$\begin{aligned}
\frac{da}{ds}(r,s) &= -\frac{v_{qb}}{v_{qa}}(z,a,b),\\
\frac{db}{ds}(r,s) &= 1, \text{ and}\\
\frac{dz}{ds}(r,s) &= 0,
\end{aligned}$$

with initial conditions

$$\begin{aligned}
a(r,s_0) &= \alpha_0(r),\\
b(r,s_0) &= \beta_0(r), \text{ and}\\
z(r,s_0) &= \phi(r).
\end{aligned}$$

To be precise, if we fix $r = r_0$, then $(a(r_0,s), b(r_0,s), z(r_0,s))$ is a characteristic curve and $(a(r_0,s), b(r_0,s))$ is a plane characteristic curve. In the text, we use both terms. We will assume that $(a(r,s), b(r,s))$ is invertible for $r \in (\underline{r}, \overline{r})$ and s such that (a,b) is in the participation region. In this case, the method provides a change of variables

$$\begin{aligned}
a &= a(r,s), \text{ and}\\
b &= b(r,s),
\end{aligned}$$

such that $q(a(r,s), b(r,s)) = \phi(r)$. Observe that we can solve explicitly for b and z,

$$\begin{aligned}
b(r,s) &= s,\\
z(r,s) &= \phi(r),
\end{aligned}$$

with $s_0 = \beta_0(r)$. It will be convenient to know the dependence of the new variable a on q. For instance, we can define $A(q,r,s)$ as the solution of

$$\frac{dA}{ds}(q,r,s) = -\frac{v_{qb}}{v_{qa}}(q,a,b)$$

with $A(q,r,s_0) = \alpha_0(r)$. Then we have

$$a(r,s) = A(\phi(r), r, s). \tag{10}$$

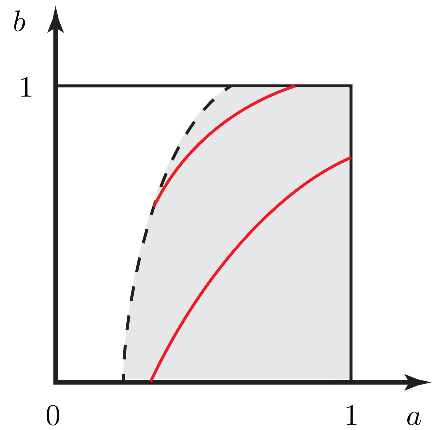

Figure 2: Two possibilities for the characteristic curves

4. Optimality Conditions

We will derive the optimality condition for the monopolist's problem. The method of characteristic curves gives a new parametrization of the type space and takes into account the local incentive condition given by Proposition 1.

In Fig. 2 we depict the two situations that may occur with the characteristic curves. The dashed curve represents the boundary between the participation (shaded) and the exclusion (white) regions. The two solid curves represent the characteristic curves. From A1 and A2, we have that the characteristic curves and the boundary curve all are increasing.

Observe that one of the curves intersects the dashed curve but the other does not. These two cases will deserve separate treatments in the following analysis.

4.1. Case I (No Intersection)

For this case, we are considering the characteristic curves that do not intersect the boundary curve between the participation and the exclusion regions.

In Fig. 3 the shaded stripes represent the image of $a(r,s)$ and $b(r,s)$ when $r \in [r_1, r_2]$ and $s \in [D(r), L(\phi(r), r)]$. The functions $D(\cdot)$ and $L(\cdot)$ give the minimum and the maximum values of the parameter s for a given r. We want to compute the contributions of these types to the monopolist's expected profit. For this task, we use the change of variables

$$\begin{cases} a(r,s) = A(\phi(r), r, s), \\ b(r,s) = s, \end{cases}$$

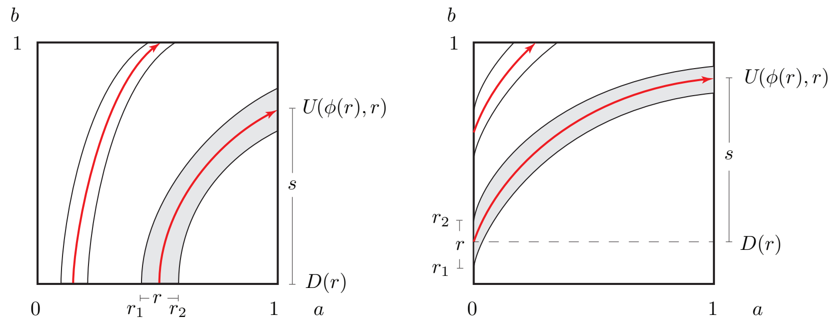

Figure 3: Illustrating the the new variables r and s.

given by the method of characteristic curves for problem (CP). Then, assuming that the Jacobian determinant is positive, i.e.,

$$\det \frac{\partial(a,b)}{\partial(r,s)} = A_q \phi' + A_r > 0,$$

we can use the change of variables formula in the integral (Π) to compute this contribution as

$$\int_{r_1}^{r_2} \int_{D(r)}^{U(\phi(r),r)} G(\phi(r), A(\phi(r), r, s), s)(A_q \phi' + A_r) ds dr. \tag{11}$$

Let us define the function

$$H(\phi, \phi', r) := \int_{D(r)}^{U(\phi,r)} G(\phi, A(\phi, r, s), s)(A_q \phi' + A_r) ds. \tag{12}$$

Then, using (12), we can rewrite (11) and define the following maximization problem

$$\max_{\phi(\cdot)} \int_{r_1}^{r_2} H(\phi(r), \phi'(r), r) dr. \tag{Π_{I}}$$

In problem (Π_{I}), we want to maximize the contribution to the monopolist's expected profit from types in the images of $a(r, s)$ and $b(r, s)$. The Euler equation for this problem is

$$H_\phi - \frac{d}{dr} H_{\phi'} = 0, \tag{13}$$

from which we get the following:

Theorem 1. *The first-order necessary condition for problem* (Π_{I}) *is given by*

$$\int_{D(r)}^{U(\phi(r),r)} \frac{G_q}{v_{qa}} (\phi(r), A(\phi(r), r, s), s) ds = 0. \tag{14}$$

Observe that Theorem 1 gives the optimality condition along the characteristic curve $\gamma(s) = (a(r,s), s)$. It is analogous to the Araujo and Moreira [1] optimality condition, now prescribed in the characteristic curve $\gamma(s) = (a(r,s), s)$.[7] The condition says that the average of the marginal virtual surplus G_q weighted by the $1/v_{qa}$ along the characteristic curve $\gamma(s)$ is zero.

4.2. Case II (Intersection)

In this case, the characteristic curve intersects the boundary between the participation and the exclusion regions. We parametrize this boundary using the curve $\beta(r)$, that we assume is differentiable when $r \in (r_1, r_2)$. For all types in this boundary, the informational rent $V(r, \beta(r)) = 0$. Then, the marginal informational rent

$$\frac{d}{dr} V(r, \beta(r)) = 0. \tag{15}$$

Using the envelope theorem from Milgrom and Segal [12] we get the following expression for marginal informational rent

$$\frac{d}{dr} V(r, \beta(r)) = v_a(\phi(r), r, \beta(r)) + v_b(\phi(r), r, \beta(r))\beta'(r). \tag{16}$$

For a simpler notation, we define the function

$$R(\phi(r), \beta(r), \beta'(r), r) := v_a(\phi(r), r, \beta(r)) + v_b(\phi(r), r, \beta(r))\beta'(r). \tag{17}$$

Therefore, equations (15) and (16) add another constraint for the monopolist's problem. Using equation (17) we can write this boundary constraint as

$$R(\phi(r), \beta(r), \beta'(r), r) = 0. \tag{BC}$$

The procedure for this case follows the procedure of the previous case. Again, in Fig. 4, the shaded stripe is the image of $a(r,s)$ and $b(r,s)$ when $r \in [r_1, r_2]$ and $s \in [\beta(r), U(\phi(r), \beta(r), r)]$. We will compute the contribution to the monopolist's expected profit from types in this shaded stripe. Now, the change of variables is represented by

$$\begin{cases} a(r,s) = A(\phi(r), \beta(r), r, s), \\ b(r,s) = s. \end{cases}$$

We also assume that the Jacobian determinant is negative,[8] i.e.,

$$\det \frac{\partial(a,b)}{\partial(r,s)} = A_q \phi' + A_\beta \beta' + A_r < 0.$$

[7] In Araujo and Moreira [1], this condition can be found in their Theorem 2 (Critical U-shaped curve).

[8] Just observe in Fig. 4 that $a(r,s)$ decreases in r. Thus the Jacobian determinant is negative.

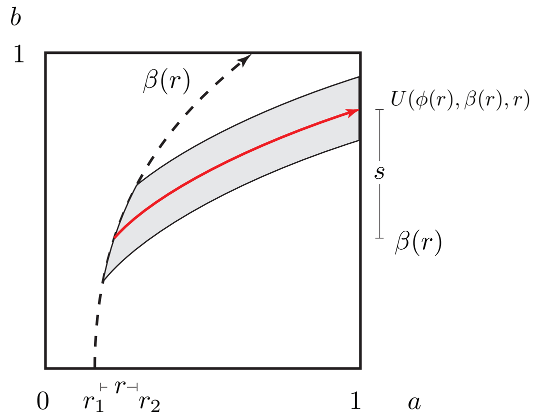

Figure 4: Illustrating the the new variables r and s.

Using the change of variables formula in the integral (Π) we can compute this contribution as

$$\int_{r_1}^{r_2}\int_{\beta(r)}^{U(\phi(r),\beta(r),r)} -G(\phi, A(\phi,\beta,r,s),s)(A_q\phi' + A_\beta\beta' + A_r)dsdr. \tag{18}$$

Defining the function

$$H(\phi,\phi',\beta,\beta',r) := \int_{\beta}^{U(\phi,\beta,r)} -G(\phi, A(\phi,\beta,r,s),s)(A_q\phi' + A_\beta\beta' + A_r)ds, \tag{19}$$

we can rewrite (18) as

$$\int_{r_1}^{r_2} H(\phi,\phi',\beta,\beta',r)dr. \tag{20}$$

We still have to consider the constraint (BC) in the monopolist's problem. For this, we use the Lagrangian multiplier $\lambda(r)$ to append this constraint. The resulting problem is

$$\max_{\phi(\cdot),\beta(\cdot)} \int_{r_1}^{r_2} \{H(\phi,\phi',\beta,\beta',r) + \lambda R(\phi,\beta,\beta',r)\}dr. \tag{Π_{II}}$$

In problem (Π_{II}) we have to optimally choose the pair $(\phi(\cdot),\beta(\cdot))$. Thus, we have a system with the two Euler equations, one for $\phi(\cdot)$ and the other for $\beta(\cdot)$.

$$\begin{cases} H_\phi - \dfrac{d}{dr}H_{\phi'} + R_\phi\lambda(r) = 0;\ \text{and,} & (21) \\[2ex] H_\beta - \dfrac{d}{dr}H_{\beta'} + \lambda(r)[R_\beta - \dfrac{d}{dr}R_{\beta'}] - R_{\beta'}\lambda'(r) = 0. & (22) \end{cases}$$

From the system of equations above we get the next:

Theorem 2. *The first-order necessary conditions for problem* (Π_{II})*, when* $R_\phi \neq 0$ *are*

$$\text{(i)} \qquad \int_{\beta(r)}^{U(\phi(r),\beta(r),r)} \frac{G_q}{v_{qa}}(\phi(r), A(\phi(r),\beta(r),r,s),s)ds = \lambda(r); \;\; \textit{and,} \tag{23}$$

$$\text{(ii)} \qquad \frac{G}{v_b}(\phi(r), r, \beta(r)) = \lambda'(r). \tag{24}$$

5. Applications

In this section we will consider the following utility function for the customer,

$$v(q,a,b) = aq - (b+c)\frac{q^2}{2}, \tag{25}$$

with $c > 0$. The monopolist's cost is assumed to be zero, therefore, we get the following virtual surplus

$$G(q,a,b) = (2a-1)q - \frac{1}{2}(b+c)q^2. \tag{26}$$

We will solve explicitly two examples, the first with $c = 1$ and the second with $c \in (0, \frac{1}{2})$.

Example 1 (Laffont et al. [10])**.** *In this example we set* $c = 1$ *for the customer's utility function* (25)*. Using the method of characteristic curves we get the following change of variables,*

$$\begin{cases} A(q,r,s) = r + sq, \\ B(q,r,s) = s. \end{cases}$$

We can determine the function $U(\cdot)$ *as*

$$U(q,r) = \begin{cases} 1, & \text{if } r < r_I, \\ \frac{1-r}{q}, & \text{if } r > r_I. \end{cases} \tag{27}$$

Using Theorem 1, the optimality condition is

$$\int_0^{U(\phi(r),r)} \frac{G_q}{v_{qa}}(\phi(r), A(\phi(r),r,s),s)ds = 0.$$

Making all the substitutions, we get

$$\int_0^U \{2(r+s\phi) - 1 - (s+1)\phi\}ds = (2r-1-\phi)U + \frac{U^2}{2}\phi = 0. \tag{28}$$

Solving equation (28) for ϕ, and using (27), we get

$$\phi(r) = \begin{cases} 0 & , \text{if } 0 \leq r \leq \frac{1}{2}, \\ 4r - 2 & , \text{if } \frac{1}{2} \leq r \leq \frac{3}{5}, \\ \frac{3r-1}{2} & , \text{if } \frac{3}{5} \leq r \leq 1. \end{cases} \tag{29}$$

Now we have to go back to the original variables. Solving the equation $A(\phi(r), r, s) = r + s\phi(r)$ for r in terms of a and b, we get:

$$r(a,b) = \begin{cases} a & , \text{if } a \leq \frac{1}{2}, \\ \\ \frac{a+2b}{1+4b} & , \text{if } \frac{1}{2} \leq \frac{a+2b}{1+4b} \leq \frac{3}{5}, \\ \\ \frac{2a+b}{2+3b} & , \text{if } \frac{3}{5} \leq \frac{2a+b}{2+3b} \leq 1. \end{cases}$$

We have that $q(a,b) = \phi(r(a,b))$. Making the substitution we can find the optimal decision $q(a,b)$:

$$q(a,b) = \begin{cases} 0 & , \text{if } a \leq \frac{1}{2}, \\ \\ \frac{4a-2}{1+4b} & , \text{if } \frac{1}{2} \leq \frac{a+2b}{1+4b} \leq \frac{3}{5}, \\ \\ \frac{3a-1}{2+3b} & , \text{if } \frac{3}{5} \leq \frac{2a+b}{2+3b} \leq 1, \end{cases}$$

and also the nonlinear tariff P that implements q:[9]

$$P(q) = \begin{cases} \frac{q}{2} - \frac{3q^2}{8} & , \text{if } q \leq \frac{2}{5}, \\ \\ \frac{q}{3} - \frac{q^2}{6} + \frac{1}{30} & , \text{if } q \geq \frac{2}{5}. \end{cases}$$

Example 2 (Deneckere and Severinov [6])**.** *Now we consider the values $c \in (0, \frac{1}{2})$ for the customer's utility function (25).*

We apply the method from Section 4 to characterize the solution proposed by Deneckere and Severinov [6], and in Fig. 5 we can see the general pattern for their solution. The participation region is divided into three subregions, $(I), (II)$ and (III).

In region (I), the characteristic curves do not intersect the boundary curve. The change of variables is given by

$$\begin{cases} A(q, r, s) = r + sq, \\ B(q, r, s) = s. \end{cases}$$

[9] We find the nonlinear tariff P by integrating the marginal tariff given by $v_q(q, a, 0)$.

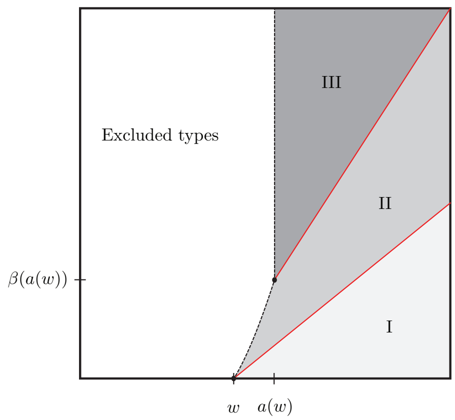

Figure 5: Solution

Then, using Theorem 1, we get the optimal[10]

$$\psi(r) = \frac{3r-1}{2c}, \text{when } r \in [w, 1]. \tag{30}$$

Observe that when $r = w$, the corresponding characteristic curve separates regions (I) and (II).

In region (II), the characteristic curves intersect the boundary curve $(r, \beta(r))$. In this case, we have the following change of variables

$$\begin{cases} A(q, \beta, r, s) = r + (s - \beta)q, \\ B(q, \beta, r, s) = s. \end{cases}$$

Solving $A(q, \beta, r, U) = 1$, we get

$$U(q, \beta, r) = \frac{1-r}{q} + \beta. \tag{31}$$

Then, using Theorem 2 we get the following system of ordinary differential equations

$$\begin{cases} \dfrac{\left(3r^2 - 4r + 1\right)\phi'(r) + 2(r-1)\phi(r)^2\beta'(r) + 2r\phi(r)}{2\phi(r)^2} = 0, \\ \\ \phi(r) - \frac{1}{2}\phi(r)^2\beta'(r) = 0. \end{cases} \tag{32}$$

---

[10] To avoid confusion, we use ψ instead of ϕ in this region I.

The solution for system (32) is given by

$$\begin{cases} \phi(r) = \dfrac{k_1}{3r^2 - 4r + 1}, \\ \\ \beta(r) = \frac{1}{k_1}(2r^3 - 4r^2 + 2r) + k_2, \end{cases} \tag{33}$$

with $r \in [w, a(w)]$. *Using the boundary conditions*

$$\psi(w) = \phi(w) \text{ and } \beta(w) = 0,$$

we get the constants k_1 *and* k_2 *depending on* w,

$$k_1 = \frac{(1-3w)^2(w-1)}{2c} \text{ and } k_2 = -\frac{4d(w-1)w}{(1-3w)^2}. \tag{34}$$

Observe that the characteristic curve that separates regions (II) *and* (III) *intersects the point* $(a(w), \beta(a(w)))$ *and the corner point* $(\alpha, \beta) = (1, 1)$. *Indeed, we define* $a(w)$ *as the solution of*

$$a + (1 - \beta(a))\phi(a) = 1. \tag{35}$$

In region (III), *all the characteristic curves intersect at the point* $(a(w), \beta(a(w)))$. *The idea is that* $q(\cdot)$ *has a discontinuity jump at* $(a(w), \beta(a(w)))$, *and this type is indifferent between all* $q \in [0, q(a(w), \beta(a(w)))]$.

Now we will compute the contribution of types in region (III) *to the monopolist's profit. We consider the following change of variables*

$$\begin{cases} A(q, s) = a(w) + (s - \beta(a(w)))q, \\ B(q, s) = s. \end{cases}$$

Using the change of variables formula we can compute this contribution as

$$\int_0^{\phi(a)} \int_{\beta(a)}^{U(q,\beta(a),a)} G(q, a(w) + (s - \beta(a(w)))q, s)(s - \beta(a(w)))dsdq. \tag{36}$$

From (11), (18), and (36) we can write the expression for the monopolist's expected profit depending on the parameter w.[11] *Finally, we have to optimally choose this* w. *We did it numerically, and we depict the result in Fig. 6, where we can see the optimal* w *for each* $c \in (0, \frac{1}{2})$.[12]

[11] See the expression for $\Pi(w)$ in Appendix B.

[12] We used the software Mathematica®, version 8.0 to perform the numeric computation for this example.

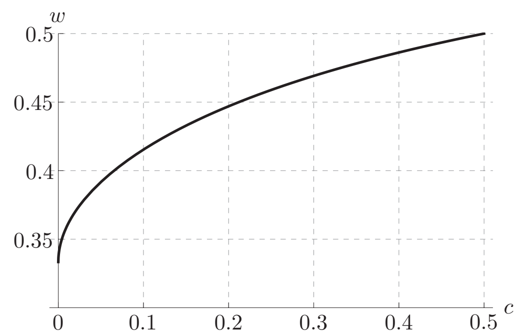

Figure 6: The optimal choice for w.

6. Conclusion

In this paper we studied a screening model where the firm is a single-product monopolist facing customers with two dimensions of heterogeneity. One can think of this paper as a natural extension of Araujo and Moreira [1] techniques to the two-dimensional context. Moreover, it also establishes a link between ideas presented in Laffont et al. [10] and Araujo and Moreira [1].

We were mainly concerned with finding the necessary optimality condition involving bunched customers, i.e., those whose types belong to the same contour line $q(a, b) = k$. After that, we used these conditions to solve concrete examples from the literature. As a further extension, we can consider how to perform these same characterizations in a more general multi-dimensional context. In other words, how to extend Theorem 1, or even Theorem 2 when the type space is multi-dimensional.

Appendix A. Mathematical Proofs

Proof of Proposition 1 The cross derivatives t_{ab} and t_{ba} are given by

$$t_{ab} = (v_{qq}(q,a,b)q_b + v_{qb}(q,a,b))q_a + v_q(q,a,b)q_{ab}, \text{ and}$$
$$t_{ba} = (v_{qq}(q,a,b)q_a + v_{qa}(q,a,b))q_b + v_q(q,a,b)q_{ba}.$$

As t is twice differentiable at (a,b), we use Schwarz's integrability condition $t_{ab} = t_{ba}$, $q_{ab} = q_{ba}$ and the result follows.

■

Proof of Theorem 1 We defined H in equation (12). For the Euler equation (13), we need to derive H_ϕ and $H_{\phi'}$. First, we derive

$$H_\phi = \int_D^U \{[G_q + G_a A_q][A_q\phi' + A_r] + G[A_{q^2}\phi' + A_{qr}]\}ds+ \tag{A.1}$$
$$+G(\phi, A(\phi,r,U),U)[A_q(\phi,r,U)\phi' + A_r]U_q.$$

Then, we derive the expression for

$$H_{\phi'} = \int_D^U GA_q ds,$$

and after that we get

$$\frac{d}{dr}H_{\phi'} = \int_D^U \{[G_q\phi' + G_a[A_q\phi' + A_r]]A_q + G[A_{q^2}\phi' + A_{qr}]\}ds+ \tag{A.2}$$
$$+G(\phi, A(\phi,r,U),U)A_q(\phi,r,U)[U_q\phi' + U_r]+$$
$$-G(\phi, A(\phi,r,D),D)A_q(\phi,r,D)D_r.$$

Therefore, using (A.1) and (A.2) we can rewrite the Euler equation as

$$\int_D^U G_q A_r ds + G(\phi, A(\phi,r,U),U)[A_r(\phi,r,U)U_q - A_q(\phi,r,U)U_r]+ \tag{A.3}$$
$$G(\phi, A(\phi,r,D),D)A_q(\phi,r,D)D_r.$$

Now we show that the expression outside the integral in (A.3) is zero. First, observe that we have two possibilities for $U(\cdot)$:

1. $U(q,r) \equiv 1$.
   In this case $U_q = 0$ and $U_r = 0$. Thus,

$$[A_r(\phi,r,U)U_q - A_q(\phi,r,U)U_r] = 0. \tag{A.4}$$

2. $A(q,r,U(q,r)) \equiv 1$.
   Differentiating with respect to q and r we get

$$\begin{cases} A_q + A_s U_q = 0, \\ A_r + A_s U_r = 0. \end{cases}$$

   Thus, we have that

$$U_q = -\frac{A_q}{A_s} \text{ and } U_r = -\frac{A_r}{A_s}.$$

   And again,

$$[A_r(\phi,r,U)U_q - A_q(\phi,r,U)U_r] =$$

$$[-A_r(\phi,r,U)\frac{A_q}{A_s} + A_q(\phi,r,U)\frac{A_r}{A_s}] = 0.$$

We also have two cases for $D(\cdot)$:

1. $D(r) \equiv 0$.
   In this case $D_r \equiv 0$ and we get

$$G(\phi, A(\phi,r,D),D)A_q(\phi,r,D)D_r \equiv 0.$$

2. $A(q,r,D(r)) \equiv 0$. In this case, for a fixed r, $A_q(q,r,D(r)) \equiv 0$ and we also get

$$G(\phi, A(\phi,r,D),D)A_q(\phi,r,D)D_r \equiv 0.$$

Therefore, we conclude that

$$G(\phi, A(\phi,r,U),U)[A_r(\phi,r,U)U_q - A_q(\phi,r,U)U_r] + \\ +G(\phi, A(\phi,r,D),D)A_q(\phi,r,D)D_r = 0,$$

and we finally get that the Euler equation is exactly

$$\int_{D(r)}^{U(\phi(r),r)} G_q(\phi(r), A(\phi(r),r,s),s)A_r(\phi(r),r,s),s)ds = 0. \tag{A.5}$$

Observe that the marginal utility is constant along the characteristic curve. Thus, we have that

$$v_q(q, A(q,r,s),s) = v_q(q, A(q,r,D(r)),D(r)).$$

Using the implicit function theorem, we get

$$A_r(q,r,s) = \frac{v_{qa}(q, A(q,r,D(r)),D(r))}{v_{qa}(q,A(q,r,s),s)}. \tag{A.6}$$

Then, plugging (A.6) into (A.5) we get the result. ■

Proof of Theorem 2(i) We now consider the Euler equation (21). We need to derive $H_\phi, H_{\phi'}$ and $\frac{d}{dr}H_{\phi'}$. First, let us derive

$$-H_\phi = \int_\beta^U \Big\{[G_q + G_a A_q][A_q\phi' + A_\beta\beta' + A_r] \tag{A.7}$$

$$+G[A_{q^2}\phi' + A_{\beta q}\beta' + A_{rq}]\Big\}ds$$

$$+G(\phi, A(\phi,\beta,r,U),U)\Big[A_q(\phi,\beta,r,U)\phi' + A_\beta(\phi,\beta,r,U)\beta' + A_r(\phi,\beta,r,U)\Big]U_q.$$

Then, we derive

$$-H_{\phi'} = \int_\beta^U GA_q ds. \tag{A.8}$$

After that, we get

$$-\frac{d}{dr}H_{\phi'} = \int_\beta^U \Big\{\big[G_q\phi' + G_a[A_q\phi' + A_\beta\beta' + A_r]\big]A_q + \tag{A.9}$$

$$+G[A_{q^2}\phi' + A_{q\beta}\beta' + A_{qr}]\Big\}ds$$

$$+G(\phi, A(\phi,\beta,r,U),U)A_q(\phi,\beta,r,U)\Big[U_q\phi' + U_\beta\beta' + U_r\Big]$$

$$-G(\phi, A(\phi,\beta,r,\beta),\beta)A_q(\phi,\beta,r,\beta)\beta'.$$

Using equations (A.7) and (A.9) we can write the Euler equation (21) as

$$-\Bigg\{\int_\beta^U \{G_q[A_\beta\beta' + A_r]\}ds + \tag{A.10}$$

$$+G(\phi, A(\phi,\beta,r,U),U)\Big[A_r(\phi,\beta,r,U)U_q - A_q(\phi,\beta,r,U)U_r\Big]$$

$$+G(\phi, A(\phi,\beta,r,U),U)\beta'\Big[A_\beta(\phi,\beta,r,U)U_q - A_q(\phi,\beta,r,U)U_\beta\Big]$$

$$+G(\phi, A(\phi,\beta,r,\beta),\beta)A_q(\phi,\beta,r,\beta)\beta' - R_\phi\lambda(r)\Bigg\} = 0.$$

We want to simplify (A.10). First observe that $A(q, \beta, r, \beta) \equiv r$. Thus, $A_q(q, \beta, r, \beta) \equiv 0$. Moreover, observe that we have two possibilities for $U(\cdot)$.

1. $U(q, \beta, r) \equiv 1$.
   In this case $U_q = U_\beta = U_r = 0$. Thus,

$$[A_r(\phi, \beta, r, U)U_q - A_q(\phi, \beta, r, U)U_r] =$$
$$[A_\beta(\phi, \beta, r, U)U_q - A_q(\phi, \beta, r, U)U_\beta] = 0.$$

2. $A(q, \beta, r, U(q, \beta, r)) \equiv 1$.
   Differentiating with respect to q, β and r we get

$$\begin{cases} A_q + A_s U_q = 0, \\ A_\beta + A_s U_\beta = 0, \\ A_r + A_s U_r = 0. \end{cases}$$

   Thus, we have that

$$U_q = -\frac{A_q}{A_s}, U_\beta = -\frac{A_\beta}{A_s} \text{ and } U_r = -\frac{A_r}{A_s}.$$

   And again,

$$[A_r(\phi, \beta, r, U)U_q - A_q(\phi, \beta, r, U)U_r] =$$

$$[-A_r(\phi, \beta, r, U)\frac{A_q}{A_s} + A_q(\phi, \beta, r, U)\frac{A_r}{A_s}] = 0,$$

   and,

$$[A_\beta(\phi, \beta, r, U)U_q - A_q(\phi, \beta, r, U)U_\beta] =$$

$$[-A_\beta(\phi, \beta, r, U)\frac{A_q}{A_s} + A_q(\phi, \beta, r, U)\frac{A_\beta}{A_s}] = 0.$$

After all these simplifications, we can write (A.10) as

$$\int_\beta^U \{G_q[A_\beta \beta' + A_r]\}ds - R_\phi \lambda(r) = 0. \tag{A.11}$$

Finally, we observe that the marginal utility is constant along the characteristic curve, i.e.,

$$v_q(q, A(q, \beta, r, s), s) = v_q(q, r, \beta). \tag{A.12}$$

Using the implicit function theorem in (A.12), we can compute

$$A_\beta(\phi(r), \beta(r), r, s) = \frac{v_{qb}(\phi(r), r, \beta(r))}{v_{qa}(\phi(r), A(\phi(r), \beta(r), r, s), s)}, \tag{A.13}$$

and

$$A_r(\phi(r), \beta(r), r, s) = \frac{v_{qa}(\phi(r), r, \beta(r))}{v_{qa}(\phi(r), A(\phi(r), \beta(r), r, s), s)}. \tag{A.14}$$

From equations (A.13) and (A.14) we can compute

$$A_\beta \beta' + A_r = \frac{R_\phi}{v_{qa}(\phi, A, s)}. \tag{A.15}$$

Using (A.15) , we can write (A.11) as

$$R_\phi[\int_\beta^U \frac{G_q}{v_{qa}}(\phi(r), A(\phi(r), \beta(r), r, s), s)ds - \lambda(r)] = 0, \tag{A.16}$$

and the result follows.

■

Proof of Theorem 2(ii) Now, we are considering equation (22). We need to derive $H_\beta, H_{\beta'}, \frac{d}{dr}H_{\beta'}, R_\beta, R_{\beta'}$ and $\frac{d}{dr}R_{\beta'}$. First, let us derive

$$\begin{aligned}
-H_\beta = \int_\beta^U & \Big\{G_a A_\beta[A_q\phi' + A_\beta\beta' + A_r] \\
& + G[A_{q\beta}\phi' + A_{\beta^2}\beta' + A_{r\beta}]\Big\}ds + \\
& + G(\phi, A(\phi, \beta, r, U), U)\Big[A_q(\phi, \beta, r, U)\phi' + A_\beta(\phi, \beta, r, U)\beta' + A_r(\phi, \beta, r, U)\Big]U_\beta \\
& - G(\phi, A(\phi, \beta, r, \beta), \beta)\Big[A_q(\phi, \beta, r, \beta)\phi' + A_\beta(\phi, \beta, r, \beta)\beta' + A_r(\phi, \beta, r, \beta)\Big].
\end{aligned} \tag{A.17}$$

Then, we derive

$$-H_{\beta'} = \int_\beta^U GA_\beta ds. \tag{A.18}$$

After that, we get

$$\begin{aligned}
-\frac{d}{dr}H_{\beta'} = \int_\beta^U & \Big\{[G_q\phi' + G_a[A_q\phi' + A_\beta\beta' + A_r]]A_\beta + \\
& +G[A_{\beta q}\phi' + A_{\beta^2}\beta' + A_{\beta r}]\Big\}ds \\
& +G(\phi, A(\phi, \beta, r, U), U)A_\beta(\phi, \beta, r, U)\Big[U_q\phi' + U_\beta\beta' + U_r\Big] \\
& -G(\phi, A(\phi, \beta, r, \beta), \beta)A_\beta(\phi, \beta, r, \beta)\beta'.
\end{aligned} \tag{A.19}$$

Using equations (A.17) and (A.19) we get

$$-H_\beta + \frac{d}{dr}H_{\beta'} = -\phi' \int_\beta^U G_q A_\beta ds + \tag{A.20}$$

$$+ G(\phi, A(\phi,\beta,r,U),U)\Big[A_q(\phi,\beta,r,U)U_\beta - A_\beta(\phi,\beta,r,U)U_q\Big]\phi'$$

$$+ G(\phi, A(\phi,\beta,r,U),U)\Big[A_r(\phi,\beta,r,U)U_\beta - A_\beta(\phi,\beta,r,U)U_r\Big]$$

$$- G(\phi, A(\phi,\beta,r,\beta),\beta)\Big[A_q(\phi,\beta,r,\beta)\phi' + A_r(\phi,\beta,r,\beta)\Big].$$

As we did in the proof of item *(i)*, we have that

$$[A_q(\phi,\beta,r,U)U_\beta - A_\beta(\phi,\beta,r,U)U_q] =$$
$$[A_r(\phi,\beta,r,U)U_\beta - A_\beta(\phi,\beta,r,U)U_r] = 0.$$

We also have that $A(\phi,\beta,r,\beta) \equiv r$, which results in $A_q(\phi,\beta,r,\beta) \equiv 0$ and $A_r(\phi,\beta,r,\beta) \equiv 1$. Therefore, we can simplify and rewrite (A.20) as

$$-H_\beta + \frac{d}{dr}H_{\beta'} = -\phi' \int_\beta^U G_q A_\beta ds - G(\phi, A(\phi,\beta,r,\beta),\beta). \tag{A.21}$$

For R, it is defined by

$$R(\phi(r),\beta(r),\beta'(r),r) = v_a(\phi(r),r,\beta(r)) + v_b(\phi(r),r,\beta(r))\beta'(r). \tag{A.22}$$

Then, we have

$$\begin{cases} R_\beta & = v_{ab}(\phi,r,\beta) + v_{b^2}(\phi,r,\beta)\beta', \\ \\ R_{\beta'} & = v_b(\phi,r,\beta), \\ \\ \frac{d}{dr}R_{\beta'} & = v_{bq}(\phi,r,\beta)\phi' + v_{ba}(\phi,r,\beta) + v_{b^2}(\phi,r,\beta)\beta'. \end{cases} \tag{A.23}$$

Finally, using the equations (A.23), (A.21), and (A.13) we can write (22) as

$$-\Bigg\{ G(\phi,r,\beta) - v_b(\phi,r,\beta)\lambda'(r) + \tag{A.24}$$

$$+\phi' v_{qb}(\phi,r,\beta)\Big[\int_\beta^U \frac{G_q}{v_{qa}}(\phi, A(\phi,\beta,r,s),s)ds - \lambda(r)\Big]\Bigg\} = 0.$$

Observe that by item *(i)*, the term in brackets in (A.24) is 0 and the result follows.

■

Appendix B. Example 2

From (11), (18), and (36) we can write the expression for the monopolist's expected profit as

$$\int_w^1\int_0^{U(\psi(r),r)} G(\psi(r), r + s\psi(r), s)(1 + s\psi'(r))dsdr+ \tag{B.1}$$

$$-\int_w^a\int_{\beta(r)}^{U(\phi(r),\beta(r),r)} \{G(\phi(r), r + (s-\beta(r))\phi(r), s)$$

$$[(s-\beta(r))\phi'(r) - \phi(r)\beta'(r) + 1]\}dsdr+$$

$$\int_0^{\phi(a)}\int_{\beta(a)}^{U(q,\beta(a),a)} G(q, a + (s-\beta(a))q, s)(s-\beta(a(w)))dsdq.$$

By an abuse of notation, we are using the same letter U for the different functions

$$U(q, r) = \frac{1-r}{q}, \tag{B.2}$$

and

$$U(q, \beta, r) = \frac{1-r}{q} + \beta. \tag{B.3}$$

Solving equation (35), we get

$$a(w) = 1 - \sqrt[3]{\frac{(w-1)\left((4c+9)w^2 - 2(2c+3)w + 1\right)}{2c}}. \tag{B.4}$$

Now, plugging the functions $\psi(r), \phi(r)$ and $\beta(r)$ from equations (30) and (33) with the constants k_1, k_2 given by (34) into the monopolist's expected profit (B.1) taking into account equations (B.2), (B.3), and (B.4) we will get an expression for the profit depending on w, given by $\Pi(w)$:

$$\Pi(w) = \frac{1}{648}\left\{\frac{27\left[-4ca(w)^3+8ca(w)^2-4ca(w)+(w-1)\left((4c+9)w^2-2(2c+3)w+1\right)\right]^2}{4c^3(a(w)-1)^3(3a(w)-1)^3}\right.$$

$$\times\left[c\left(24a(w)^3-42a(w)^2+24a(w)+3w^3-9w^2+5w-5\right)+\right.$$
$$\left.+2(w-1)(1-3w)^2+2(w-1)(1-3w)^2\right]+$$

$$+3\left[3w(3w(3w-5)+11)-16\log(-1+3w)-15+16\log(2)\right]$$

$$-\frac{18\left[-\left(9w^3+3w+10\right)a(w)+9a(w)^4-18a(w)^3+24a(w)^2+w\left(18w^2-21w+10\right)\right]}{3a(w)-1}$$

$$\left.-\frac{6\left[8(3a(w)-1)\log(1-3w)+(8-24a(w))\log(1-3a(w))\right]}{3a(w)-1}\right\}.$$

Then, for each $c\in(0,\frac{1}{2})$ we maximize this expression numerically and we get Fig. 6.

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