Dynamics converging to optimal transport

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Optimal transportation on the round sphere

- Source domain: $M = S^n = \bar{M}$: Target domain
- Transportation cost $c(x, \bar{x}) = \frac{1}{2}\mathrm{dist}^2(x, \bar{x})$.
- $\rho(x)$: source mass distribution
  $\bar{\rho}(\bar{x})$: target mass distribution
  $0 < \rho, \bar{\rho} \in C^\infty$. $\int_M \rho d\,\mathrm{vol}_M = \int_{\bar{M}} \bar{\rho} d\,\mathrm{vol}_{\bar{M}}$
- Transportation map $F : M \to \bar{M}$, Borel measurable
  $F_\# \rho = \bar{\rho}$, i.e., $\rho(F^{-1}(\Omega)) = \bar{\rho}(\Omega)$ for all Borel $\Omega \subset \bar{M}$.

Question: Find and describe the optimal transportation

$$\underset{F_\# \rho = \bar{\rho}}{argmin} \int_M c(x, F(x)) \rho(x) d\,\mathrm{vol}_M(x)$$

Optimal maps

Theorem (Monge, Kantorovich, $\cdots\cdots\cdots$, Brenier, McCann, $\cdots\cdots$, Ma, Trudinger, Wang, Loeper, $\cdots$)

- $\exists!$ *Optimal Map* T
- $T = T_\phi(x) = \exp_x \nabla\phi(x)$ *for the c-***potential** $\phi \in C^\infty(M, \mathbf{R})$ *which is strictly c-convex:*
  i.e. $D^2_{xx}\phi + D^2_{xx}c(x, T_\phi(x)) > 0$ *for all* $x \in M$.

PDE: Monge-Ampère type equations for mass transport

$T_{\#}\rho = \bar{\rho}$ & c-potential ϕ $\Longrightarrow$ " Monge-Ampère type equation"

$$|dT(x)| = \frac{\rho(x)}{\bar{\rho}(T(x))}$$

$$T_{\#}\rho = \bar{\rho} \Longrightarrow \det\left(D^2_{xx}\phi(x) + D^2_{xx}c(x, T_\phi(x))\right) = \left|D^2_{x\bar{x}}c(x, T_\phi(x))\right| \frac{\rho(x)}{\bar{\rho}(T_\phi(x))}.$$

Dynamics: heat flow for optimal transport

$$\frac{\partial}{\partial t} u = \theta(x, u)$$

where

$$\theta(x, u) = \log \det(D^2_{xx} u(x) + D^2_{xx} c(x, T_u(x))) - \log |\det(-D_x D_{\bar{x}} c(x, T_u(x)))| - \log \rho(x) + \log \bar{\rho}(T_u(x))$$

and

$$T_u(x) := \exp_x \nabla u(x)$$

Note: $\theta = 0 \implies T_{\#}\rho = \bar{\rho}$ and T becomes optimal.

Question: Starting from the identity map $T_0(x) = x$, (i.e. $u_0 \equiv 0$), does the flow exist and converge to the optimal map?
rmk: $(T_0)_{\#}\rho \neq \bar{\rho}$.

Why interesting?

- It may describe the dynamics to optimal transport.
- It gives **an algorithm for finding the optimal map**.
- It may give an alternative way to show smoothness of optimal map (via parabolic smoothing).

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Theorem (K., Streets & Warren)

Yes. *Moreover, the convergence is* **exponentially** *fast! That is,*

$$\|T_u - T_\phi\|_{C^k} \lesssim e^{-\delta_k t}, \text{ for all large } t > 0, \qquad k = 0, 1, 2, \cdots$$

Rmk:
[Kitagawa] obtained similar results on "good" domains in $\mathbf{R}^n$ with "good" costs.

How to show?

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A serious obstacle: On **global** manifold domains $M, \bar{M}$, $c \notin C^2_{loc}$ on **cut locus**.

- On S^n. $|D_x \mathrm{dist}^2(x, \bar{x})| \to \infty$ as $x \to -\bar{x}$. $\mathrm{Cut}(\bar{x}) = -\bar{x}$.

KEY for global manifold domains:
show for maps $T_u(x) = \exp_x \nabla u(x)$,

if $\gamma \leq |\det DT_u| \leq \frac{1}{\gamma}$,

then T_u **"Stay Away from cut locus"**.

i.e. $\mathrm{dist}(T_u(x), \mathrm{Cut}(x)) \geq \delta(\gamma, M) > 0, \forall x \in M$.

This is not known in general case.

But, holds for

- $M = \bar{M} = S^n$, $c = \mathrm{dist}^2/2$. [Delanoë & Loeper]
- $M = \bar{M} = S^{n_1}_{r_1} \times \cdots \times S^{n_k}_{r_k}$, $c = \mathrm{dist}^2/2$. [Figalli, K. & McCann]
- $\cdots$

How to show?

$$\frac{\partial}{\partial t} u = \theta(x, u)$$

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Linearized operator

To show $\gamma \leq |\det DT_u| \leq \frac{1}{\gamma}$ (so that "stay away from cut locus" holds), use maximum principle for the linearized operator.

$$\frac{\partial}{\partial t} u = \theta(x, u)$$

$$\theta(x,u) = \log\det(D^2_{xx}u(x) + D^2_{xx}c(x, T_u(x)))$$
$$- \log|\det(-D_x D_{\bar{x}} c(x, T_u(x)))| - \log\rho(x) + \log\bar{\rho}(T_u(x))$$

Let $w_{ij}(x) = D^2_{x_i x_j} u(x) + D^2_{x_i x_j} c(x, T_u(x))$

L: linearized operator of $u \to \theta(x,u)$.

$$Lv := \frac{d}{ds}\Big|_{s=0} \theta(x, u+sv) = w^{ij} D^2_{x_i x_j} v - [\cdots] D_{x_k} v$$

Note: L has no zeroth order term. $\Longrightarrow$ elliptic/parabolic maximum principle

Note:

$$\frac{\partial}{\partial t}\theta = L\theta$$

"maximum principle" $\Longrightarrow$ "θ is bounded"
$\Longrightarrow$ "$\log|\det DT_u| \leq C$"
$\Longrightarrow$ **"T_u stays away from cut locus"**

key inequality

$$w_{ij}(x) = D^2_{x_i x_j} u(x) + D^2_{x_i x_j} c(x, T_u(x))$$

Lemma

Let $\{\frac{\partial}{\partial x_i}\}$ be orthonormal basis. Then,

$$\frac{\partial}{\partial t} w_{11} - L w_{11} \le -\sum_i w^{ii} \mathbf{MTW}_i^1 w_{11}^2 + C(\rho, \bar{\rho})(1 + \sum_i w_{ii}^2 + \sum_i w^{ii} \sum_i w_{ii})$$

Here, **MTW** *is the Ma-Trudinger-Wang curvature tensor.*

- **MTW** > 0 *on S^n. [Loeper]*

Apply **maximum principle** and **stay away from cut locus**,
"**Lemma**" $\Longrightarrow$ " $|w_{ij}| \le C$"
$\Longrightarrow$ " $C_1 \le w_{ij} \le C_2$" $\Longrightarrow$ " L and θ are uniformly elliptic"
$\Longrightarrow\Longrightarrow$ **"the exponential convergence to optimal map"**.

Ma-Trudinger-Wang curvature

Definition (Ma-Trudinger-Wang curvature)

$A(x,p) = -D^2_{xx}c(x, \exp_x p)$

$$\mathbf{MTW}^\eta_\xi = D^2_{p_\eta p_\eta}[A(x,p)x_\xi x_\xi] = -c^{\bar{k}a}c^{\bar{l}b}[c_{ij\bar{k}\bar{l}} + c_{ij\bar{n}}c^{\bar{n}m}c_{m\bar{k}\bar{l}}]\xi^i\xi^j\eta^a\eta^b$$

- Trivial: For $c = |x-y|^2/2$ on $\mathbf{R}^n$, **MTW** $\equiv 0$
- Some examples due to [Ma, Trudinger, Wang]. e.g. **MTW** > 0 for $c = \log|x-y|$ on $S^n \subset \mathbf{R}^{n+1}$
- [Loeper] For $c = dist^2$ on S^n, **MTW** > 0.
- other examples due to [K. & McCann], [Figalli, Rifford & Villani] , [Delanoë & Ge], $\cdots$

Rmk: Verifying **MTW** condition is difficult.
Rmk: **MTW** ≥ 0 is related to continuity/smoothness of optimal transport as well as the geometry of the domain:

Thank You Very Much!