Multi-marginal optimal transportation and hedonic pricing

Brendan Pass

University of Alberta

June 4, 2012

Introduction

- Probability measures μ_i on $X_i \subseteq \mathbb{R}^n$, $i = 1, 2, ...m$.
  - distribution of *types* i.
- Space $Z \subseteq \mathbb{R}^n$ of *contracts*.
- *Utility* functions $f_i : X_i \times Z \to \mathbb{R}$.
  - $f_i(x_i, z) =$ preference of type $x_i \in X_i$ for contract $z \in Z$.

Equilibrium: two formulations

Carlier-Ekeland (2010): Look for a **measure** ν on Z and **couplings** π_i of μ_i and ν maximizing:

$$\sum_{i=1}^{m} \int_{X_i \times Z} f_i(x_i, z) d\pi_i$$

Proved:

1. *Uniqueness* when $\mu_1 << dx_1$ and $z \mapsto D_{x_1} f_1(x_1, z)$ is injective.
2. *Purity* when $\mu_i << dx_i$ and $z \mapsto D_{x_i} f_i(x_i, z)$ is injective.

Question: What does ν look like?

Equilibrium: two formulations

Carlier-Ekeland (2010): Look for a **measure** ν on Z and **couplings** π_i of μ_i and ν maximizing:

$$\sum_{i=1}^{m} \int_{X_i \times Z} f_i(x_i, z) d\pi_i$$

Proved:

1. *Uniqueness* when $\mu_1 << dx_1$ and $z \mapsto D_{x_1} f_1(x_1, z)$ is injective.
2. *Purity* when $\mu_i << dx_i$ and $z \mapsto D_{x_i} f_i(x_i, z)$ is injective.

Question: What does ν look like?

Chiappori-McCann-Nesheim (2010): Set $b(x_1, x_2, ..., x_m) = \max_z \sum_{i=1}^{m} f_i(x_i, z)$ and look for a **coupling** γ of $\mu_1, \mu_2, ..., \mu_m$ maximizing:

$$\int_{X_2 \times X_2 \times ... \times X_m} b(x_1, x_2, ..., x_m) d\gamma,$$

A multi-marginal optimal transportation problem.

Questions: Uniqueness? Structure? Purity?

A theorem

Theorem (P 2011)

Assume

1. *For all i, f_i is C^2 and the matrix $D^2_{x_i z} f_i$ of mixed, second order partial derivatives is everywhere non-singular.*
2. *For each $(x_1, x_2, ..., x_m)$ the maximum is attained by a unique $z(x_1, x_2, ..., x_m) \in Z$.*
3. *$\sum_{i=1}^{m} D^2_{zz} f_i(x_i, z(x_1, x_2, ..., x_m))$ is non-singular.*

$\Rightarrow$ $spt(\gamma)$ is contained in an n-dimensional, Lipschitz submanifold $S \subseteq X_1 \times X_2 \times ... \times X_m \subseteq \mathbb{R}^{nm}$.

- Special case of a more general result.
- Hedonic pricing surpluses b work very nicely here.

Geometry of spt(γ)

Set

$$M = \begin{bmatrix} 0 & D^2_{x_1x_2}b & D^2_{x_1x_3}b & \dots & D^2_{x_1x_m}b \\ D^2_{x_2x_1}b & 0 & D^2_{x_2x_3}b & \dots & D^2_{x_2x_m}b \\ D^2_{x_3x_1}b & D^2_{x_3x_2}b & 0 & \dots & D^2_{x_3x_m}b \\ \dots & \dots & \dots & \dots & \dots \\ D^2_{x_mx_1}b & D^2_{x_mx_2}b & D^2_{x_mx_3}b & \dots & 0 \end{bmatrix}$$

- A symmetric, $(nm) \times (nm)$ matrix.
- Signature:
  - n-positive eigenvalues.
  - $m(n-1)$-negative eigenvalues.
- spt(γ) is *spacelike*:
  - $\vec{x}^T M \vec{x} \geq 0$ for all tangent vectors $\vec{x} \in X_1 \times X_2 \times \dots \times X_m$ to spt(γ).

Work in Progress

Understanding the geometry of spt(γ) lets us prove things about ν:

Theorem (P 2012)

ν is absolutely continuous with respect to Lebesgue measure.

More work in progress

This, in turn, lets us prove more things about γ:

Theorem (P 2012)

Assume

1. $\mu_1 << dx_1$.
2. $z \mapsto D_{x_1} f_1(x_1, z)$ *is injective.*
3. $x_i \mapsto D_z f_i(x_i, z)$ *is injective.*

$\Rightarrow \gamma$ *is unique and pure.*