Two-dimensional screening: a useful optimality condition

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Outline

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- The customers' preferences exhibit two dimensions of uncertainty, which are their private information.
- We will derive the necessary optimality conditions characterizing the level curves for the optimal quantity assignment function.

Model

- The customer has a quasi-linear preference

$$v(q, a, b) - t,$$

where $(a, b) \in \Theta = [0, 1] \times [0, 1]$ is the customer's type, $q \in \mathbb{R}_+$ is the good consumed, and t is the monetary payment.

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- The firm does not observe (a, b) and has a prior distribution over Θ according to the differentiable density function $f(a, b) > 0$.
- The monopolist's preference is given by

$$\Pi(q, t) = t - C(q),$$

where $C(q)$ is a the cost function.

Monopolist's Problem

Formally, the monopolist's problem consists in choosing the contract $(q, t) : \Theta \to \mathbb{R}_+ \times \mathbb{R}$ that solves

$$\max_{q(\cdot),t(\cdot)} \int_0^1 \int_0^1 \Pi(q(a,b), t(a,b)) f(a,b) da db, \tag{Π}$$

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and the *incentive compatibility* constraints:

$$(a,b) \in \arg \max_{(a',b') \in \Theta} \{v(q(a',b'), a, b) - t(a',b')\}, \quad \forall (a,b) \in \Theta. \tag{IC}$$

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2. Using the method of characteristic curves, we get a new and convenient parametrization of the type space.
3. In these new variables, we derive the optimality condition for the monopolist's problem along the characteristic curve.

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we get the following partial differential equation (PDE):

$$-\frac{v_{qb}}{v_{qa}} q_a + q_b = 0. \quad (3)$$

Cauchy Initial Value Problem

$$\begin{cases} -\dfrac{v_{qb}}{v_{qa}} q_a + q_b = 0, \\ \\ q_{|_\Gamma} = \phi(r). \end{cases} \qquad \text{(CP)}$$

- The idea is to prescribe the value of $q(\cdot)$ on Γ and then use the characteristic curves to propagate this information to the participation region.
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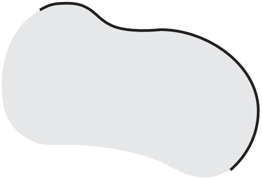

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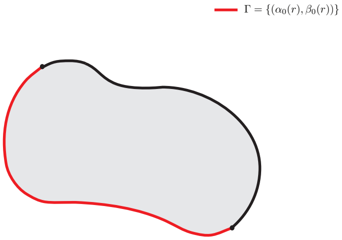

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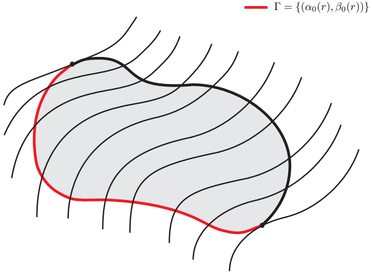

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Characteristic Curves

- We define the family of curves $(a(r,s), b(r,s), z(r,s))$ as the solution of

$$\frac{da}{ds}(r,s) = -\frac{v_{qb}}{v_{qa}}(z,a,b),$$
$$\frac{db}{ds}(r,s) = 1, \text{ and}$$
$$\frac{dz}{ds}(r,s) = 0,$$

with initial conditions

$$a(r,s_0) = \alpha_0(r),$$
$$b(r,s_0) = \beta_0(r), \text{ and}$$
$$z(r,s_0) = \phi(r).$$

Change of Variables

- 'Solving' the system, we get

$$
\begin{aligned}
a(r,s) &= A(\phi(r), r, s), \\
b(r,s) &= s, \\
z(r,s) &= \phi(r),
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Change of Variables

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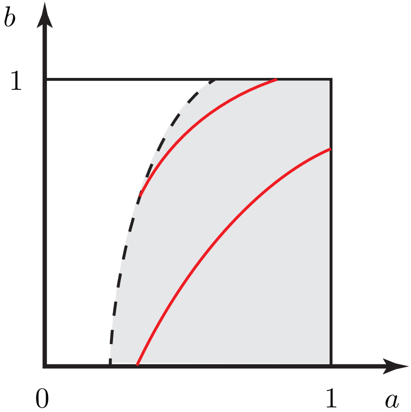
$$a(r,s) = A(\phi(r), r, s),$$
$$b(r,s) = s,$$
$$z(r,s) = \phi(r),$$

where $A(q,r,s)$ is the solution of

$$\frac{dA}{ds}(q,r,s) = -\frac{v_{qb}}{v_{qa}}(q,a,b)$$

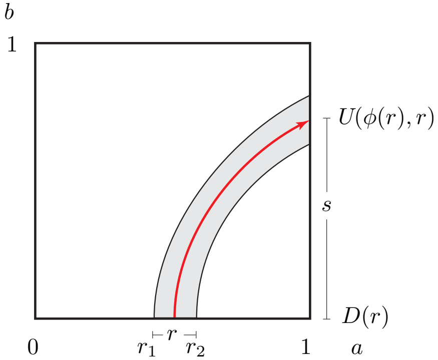
with $A(q,r,s_0) = \alpha_0(r)$.

Two Possibilities



Two possibilities for the characteristic curves

1-No Intersection



Illustrating the the new variables r and s.

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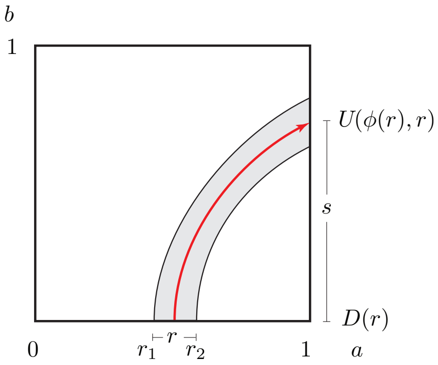

Illustrating the the new variables r and s.

- We want to compute the contribution of these types to the monopolist's expected profit.

Expected Profit

- First we assume that we already eliminated the monetary transfer from the monopolist's expected profit, and the new expression is

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we can compute this contribution as

$$\int_{r_1}^{r_2} \int_{D(r)}^{U(\phi(r),r)} G(\phi(r),A(\phi(r),r,s),s)(A_q\phi' + A_r)dsdr. \tag{4}$$

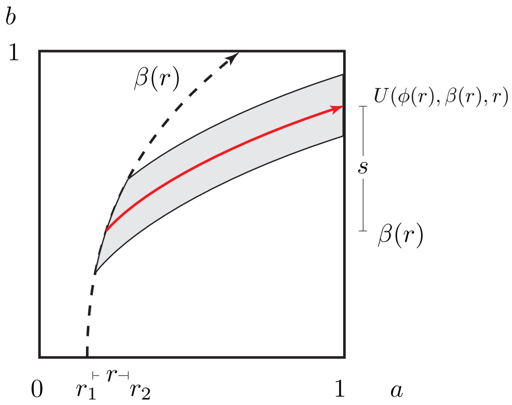
Optimality Condition

Theorem 1

The first-order necessary condition for $\phi(r)$ is given by

$$\int_{D(r)}^{U(\phi(r),r)} \frac{G_q}{v_{qa}}(\phi(r), A(\phi(r), r, s), s)ds = 0.$$

2-Intersection



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- So we have also to consider the constraint

$$R(\phi(r), \beta(r), \beta'(r), r) := v_a(\phi(r), r, \beta(r)) + v_b(\phi(r), r, \beta(r))\beta'(r) = 0.$$

Expected Profit

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and we can compute this contribution as

$$\int_{r_1}^{r_2} \int_{\beta(r)}^{U(\phi(r),\beta(r),r)} -G(\phi, A(\phi,\beta,r,s), s)(A_q \phi' + A_\beta \beta' + A_r) ds dr.$$

Optimality Conditions

Theorem 2

The first-order necessary conditions for $\phi(r)$ and $\beta(r)$, when $R_\phi \neq 0$ are

(i) $$\int_{\beta(r)}^{U(\phi(r),\beta(r),r)} \frac{G_q}{v_{qa}}(\phi(r), A(\phi(r), \beta(r), r, s), s)ds = \lambda(r);$$ *and,*

(ii) $$\frac{G}{v_b}(\phi(r), r, \beta(r)) = \lambda'(r).$$

Example

We consider the following utility function,

$$v(q, a, b) = aq - (b + c)\frac{q^2}{2},$$

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- When $c \in (0, \frac{1}{2})$, we have to apply Theorems 1 and 2. (This is Deneckere and Severinov (201?) example.)

The End

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- We can determine the function $U(\cdot)$ as

$$U(q, r) = \begin{cases} 1, & \text{if } r < r_I, \\ \frac{1-r}{q}, & \text{if } r > r_I. \end{cases} \qquad (5)$$

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- Solving equation (6) for ϕ, and using (5), we get

$$\phi(r) = \begin{cases} 0 & \text{, if } 0 \leq r \leq \frac{1}{2}, \\ 4r - 2 & \text{, if } \frac{1}{2} \leq r \leq \frac{3}{5}, \\ \frac{3r-1}{2} & \text{, if } \frac{3}{5} \leq r \leq 1. \end{cases} \qquad (7)$$