Equal Treatment as a Second Best: Student Loans under Asymmetric Information*

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15 May 2012

Abstract

We study the set of second-best optimal "menus" of student-loan contracts in a simple economy with risky labour-market outcomes, adverse selection, moral hazard and risk aversion. There are two categories of second-best optima: the *equal treatment* and the *separating* allocations. The equal treatment case is obtained when the social weights of student types are close to their population frequencies; the students' *ex post* payoffs are then the same for each type as a function of the random individual outcome. In separating optima, the talented types bear more risk than the less-talented ones; they arise only if the weight of the talented types is sufficiently high. Second-best optimal loan contracts are always income-contingent; they provide incomplete insurance; they typically involve cross-subsidies. If viewed as a graduate tax, the second-best optimal repayment schedule is progressive.

*Preliminary version. Comments are welcome.

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1 Introduction

The importance of student loans for the accumulation of human capital, economic growth and welfare cannot be overestimated. In the United States, the total amount of outstanding student debt has reached \$1 trillion at the end of 2011. In Great Britain, the rise of tuition fees seems to have caused a sharp increase in average student debt[1]. With the recent economic downturn, it became clear that an increasing number of students experience difficulties to repay their loans[2]. Student loans pose interesting financial engineering and regulation problems. There are many discussions on the optimal design of these loans: for instance, the UK and Australia have a form a income-contingent repayment system, since loan repayments are based on the graduate's monthly earnings, just like income tax, and interest rates are subsidized[3]. In some continental European countries, student loans play a negligible *role* but, given the severe shortage of public funds, they could go hand in hand with a substantial raise in tuition fees, and become a new source of funds for universities[4]. There is an important econometric literature on the impact of credit constraints on university or college attendance[5]. For recent quantitative studies of alternative student-loan policies in the US, see, *e.g.*, Ionescu (2009), Lochner and Monge-Naranjo (2010).

These questions are hotly debated, yet, to the best of our knowledge, the microeconomic theory of student loans is still underdeveloped[6]. In particular, we need a normative foundation for the intuition that income-contingent loans are the appropriate solution, when informational asymmetries between lenders and borrowers are involved in the allocation and design of loans. In the following, we propose a simple model of student loans, under the combined effects of risk aversion, moral hazard and adverse selection. We explore the structure of the set of second-best optimal (or interim incentive-efficient) allocations of credit

[1]The average student debt is predicted to be around 50,000 pounds, on leaving the university, for those starting in 2012. for details, see *http://www.slc.co.uk/statistics*.

[2]See, for instance, *The Economist*, October 29th, 2011, p17 and P 73. In the US and in 2009, the default rate on student loans has reached 8.8%.

[3]See, *e.g.*, Barr and Johnston (2010).

[4]See, *e.g.*, Jacobs and Van der Ploeg (2006).

[5]See, for instance, Carneiro and Heckman (2002), Keane and Wolpin (2001), Stinebrickner and Stinebrickner (2008). For a survey, see Lochner and Monge-Naranjo (2011).

[6]See our discussion of the literature below.

to risk-averse students in an economy in which individual talents and efforts are not observed by the lender, and future earnings are subject to risk.

Our main results are the following. We consider an economy with two unobservable types of students, the talented (or low-risk) and the less-talented (high-risk) students, where risk affects earnings. On the labour market, the hard-working (high-effort) talented types earn a high wage with a higher probability than the low-effort or the less-talented types. We describe the set of second-best Pareto-optima by letting the social weight of types vary in the social welfare function (*i.e.*, a standard weighted average of utilities).

There are two broad categories of second-best optima, namely, the *separating* and *equal treatment* optima. When the social weight of types is in the neighborhood of their frequencies in the population of students, and therefore, in the vicinity of the standard utilitarian case, the second-best optimal menu of contracts exhibits a form of *pooling*, called *equal treatment*: the students' *ex post* payoffs, net of loan repayments, are the same as a function of the random individual outcome. In other words, net earnings as a function of individual "success" or "failure" on the labour market should be independent of the student's type. But of course, in spite of being treated equally in this particular sense, students are *ex ante* unequal, since the talented types have a greater probability of success. This first type of solution is also characterized by *bunching* in the sense that it remains constant as a function of social weights on an interval. This is not the allocation described in textbook treatments of models of insurance under adverse selection à la Rothschild-Stiglitz[7]. To obtain the familiar *separating* menu of loan contracts as a second-best optimum, we need to increase the weight of the talented type relative to its natural frequency in the population. Hence, in the vicinity of the standard utilitarian case, equal treatment (as defined above) is incentive compatible and second-best optimal.

The optimal menu of contracts exhibits *incomplete insurance*: this is mainly due to moral hazard. In the case of a separating optimum, both types are incompletely insured but the talented types bear more risk than the less-talented. The less-talented obtain the maximal amount of income smoothing compatible with incentives to exert high effort. The same incomplete insurance property also emerges in the pure adverse-selection case, when

[7]See Rothschild and Stiglitz (1976).

repayments on loans are constrained to be non-negative.

As a by-product, we find that *second-best optimal contracts are always income-contingent*. Finally, the budget is by construction balanced (we did not explore interest-rate subsidies that would be financed by means of external sources of funds) but the second-best optima typically exhibit *cross-subsidies* between types: the talented repay more and subsidise the less-talented. If interpreted as a *graduate tax*, the second-best solution is progressive.

It is well-known that microeconomic models of insurance and models of banking are formally close. Rothschild and Stiglitz's approach to screening in insurance markets has been applied to banking, albeit with adaptations (see, *e.g.*, Bester (1985)). Classic theories of credit contracts typically treat adverse selection and moral hazard separately (see Freixas and Rochet (1998)). A contribution of the present paper is to propose a study of the structure of second-best optima in a screening model à la Rothschild-Stiglitz, but with the added complication of moral hazard, since outcome probabilities also depend on hidden actions[8]. Student loans are a very natural application for the theory of incentives or Mechanism Design under hidden actions and hidden types. The general theory of optimum (or equilibrium) contracts under moral hazard, adverse selection *and* risk aversion is known to be a very hard problem (see Arnott (1991) for comments and further references to unpublished essays on this question[9]). Solutions can be exhibited when principal and agent are both risk-neutral (see, *e.g.*, Picard (1987) and Caillaud, Guesnerie and Rey (1992), see also the discussion in Laffont and Martimort (2002, chapter 7)). In the field of optimal regulation theory, a few contributions have dealt with special cases (see, *e.g.*, McAfee and McMillan (1986), Baron and Besanko (1987), Laffont and Rochet (1998)). An extension of Rothschild and Stiglitz's insurance market model to moral hazard, and hence the study of equilibria in such an extended model, is proposed in the often quoted, but unpublished manuscript of Chassagon and Chiappori (1997). Our model is very close to that of the latter contribution, but Chassagnon and Chiappori did not study cross-subsidies between types and the set of Pareto optima. Recent work on the Principal-Agent model in the case at hand required advanced mathematical optimisation techniques (see Faynzilberg and Kumar (2000)) or used

[8]The structure of second-best optima in insurance markets with pure adverse selection has been studied by Crocker and Snow (1985) and Henriet and Rochet (1990).

[9]For an early attempt, see Dionne and Lasserre (1985).

stochastic calculus, as in the asset-pricing, continuous-time finance literature (see, *e.g.*, Sung Jaeyoung (2005)). These intimidating technicalities mainly explain why we study a simple textbook model here, but it conveys, we think, the essential intuitions and ideas (and yet, some of the proofs are not straightforward). Finally, Chatterjee and Ionescu (2011) propose a quantitative analysis of a model of student loans with moral hazard, exploring the feasibility of offering insurance against college-failure risk. But they do not treat adverse selection as in the present paper, (*i.e.*, in contrast to our approach, which is standard in the Mechanism Design literature, Chatterjee and Ionescu (2011) rely neither on *menus of contracts* nor on the *revelation principle*).

In the following, Section 2 describes the model, states basic assumptions, and studies first-best optima. Section 3 is devoted to a preliminary analysis of the pure adverse selection case. This is done first when cross-subsidies between student types are not permitted, yielding a model *à la* Rothschild-Stiglitz. In this case, separating menus of loan contracts are optimal. We then consider the pure adverse selection case with cross-subsidies, but under a non-negative repayment constraint: the banker cannot also become an insurer. In this case, we find the equal treatment and incomplete insurance result described above, even if moral hazard plays no role. The result is due to the impossibility of negative loan repayments in case of failure. Separating optima arise only if the social weight of talented types is high enough. Finally, Section 4 presents the relatively more difficult case in which adverse selection and moral hazard are combined, and we find analogous results: the equal treatment and incomplete insurance results are obtained when the social weights of types are close to their respective population frequencies; separating menus of contracts are optimal only if the social weight of the talented types is high enough.

2 A Simple Model

2.1 Basic Assumptions

We consider a population of students with the same von Neumann-Morgenstern utility $u(.)$, assumed differentiable. There are two types of students, indexed by $i = 1, 2$. The types have different probabilities of success, denoted p_i.

We assume that in case of "success", *i.e.*, with probability p_i, type i obtains a wage $w(q)$ on the labor market if he or she completed an education of quality q. In the case of "failure", *i.e.*, with probability $1 - p_i$, the student gets the wage w_0. We assume $0 < p_i < 1$; the total cost of education is simply $\gamma_i q$ for quality q; the unit cost γ is positive and depends on type; w is a continuously differentiable and strictly concave function of the nonnegative real number q. In addition, we assume the following.

Assumption 1.

a) $u(.)$ is strictly increasing and strictly concave.

b) $w(q) \geq w_0$ for all $q \geq 0$.

c) $p_2 > p_1$.

d) $\gamma_1 \geq \gamma_2$

All students are risk-averse. The event of success on the labour market yields a wage always greater than w_0, and the type 2 are the "high types": they are at the same time more likely to succeed and cheaper to educate.

A student loan is always covering the cost of education. So, the magnitude of a loan to type i is $\gamma_i q_i$ for an education of quality q_i. Let (R_i, r_i) denote the reimbursement profile. Reimbursement is contingent on success. A type i student gets the income $w(q_i) - R_i$ with probability p_i and the income $w_0 - r_i$ with probability $1 - p_i$.

We will study cross-subsidies between student types. Let λ_i denote the frequency of type i in the student population (we have $\lambda_1 + \lambda_2 = 1$). Assume that a public lending authority distributes all loans; this public banker's *per capita* resource constraint imposes,

$$\Sigma_i \lambda_i (p_i R_i + (1 - p_i) r_i) \geq \Sigma_i \lambda_i \gamma_i q_i. \tag{1}$$

Let t_i denote the *per capita* subsidy of type i. By definition, we have,

$$t_i = p_i R_i + (1 - p_i) r_i - \gamma_i q_i \tag{2}$$

This variable must be interpreted as a tax if $t_i > 0$ and as a proper subsidy if $t_i < 0$. The resource constraint is equivalent to $\Sigma_i \lambda_i t_i \geq 0$. By definition, in this economy, an allocation, (or a menu of contracts) is an array $\{(q_i, R_i, r_i)\}_{i=1,2}$.

2.2 First-Best Optimality

Denote $V_i = p_i U_i + (1-p_i)u_i$, the expected utility of type i, where, by definition

$$U_i = u(w(q_i) - R_i) \qquad \text{and} \qquad u_i = u(w_0 - r_i). \tag{3}$$

Let α_1 and α_2 be the weights of type 1 and type 2 in the welfare function. A first-best optimum can be obtained as a solution of the following problem,

$$\text{Maximize} \quad \alpha_1 V_1 + \alpha_2 V_2 \tag{4}$$

subject to,

$$\Sigma_i \lambda_i [p_i R_i + (1-p_i) r_i - \gamma_i q_i] \geq 0. \tag{5}$$

Define $z(x) = u^{-1}(x)$, the inverse utility function. We get

$$z(U_i) = w(q_i) - R_i \qquad \text{and} \qquad z(u_i) = w_0 - r_i. \tag{6}$$

We these definitions, the first-best optimality problem can be rewritten as follows. Eliminating R_i and r_i from the problem and the above resource constraint, we obtain

$$\text{Maximize} \quad \sum_i \alpha_i (p_i U_i + (1-p_i) u_i) \tag{7}$$

with respect to $(q_i, U_i, u_i)_{i=1,2}$ subject to the resource constraint,

$$\sum_i \lambda_i B_i(q_i) \geq \sum_i \lambda_i (p_i z(U_i) + (1-p_i) z(u_i)), \tag{$\overline{RC}$}$$

where, by definition,

$$B_i(q_i) = p_i w(q_i) + (1-p_i) w_0 - \gamma_i q_i. \tag{8}$$

The function $B_i(q_i)$ is the expected surplus of higher education for type i.

It is now easy to show that the efficient choice of q_i must maximize $B_i(q_i)$ for all $i = 1, 2$. We necessarily have $q_i = q_i^*$, where

$$p_i w'(q_i^*) = \gamma_i, \tag{9}$$

and where w' denotes the derivative of w. This condition is necessary and sufficient since w is concave, an it is also easy to check that $B_2(q_2^*) > B_1(q_1^*)$, since $p_2 > p_1$ and $\gamma_1 \geq \gamma_2$. We assume that the first-best education is interior and that the efficient amount of education is profitable, on average, for both types. More precisely, we assume the following.

Assumption 2. $q_i^* > 0$, $i = 1, 2$; and $B_1(q_1^*) > w_0$.

The first-best problem is a convex problem, since $z(.)$ is a convex function and the objective is a linear function of utility levels U_i, u_i. To write the first-order necessary conditions for optimality, let β denote the Lagrange multiplier of the resource constraint. Standard computations yield,

$$\beta\lambda_i B_i'(q_i) = 0 \qquad (10)$$

$$\alpha_i p_i = \beta\lambda_i p_i z'(U_i) \qquad (11)$$

$$\alpha_i(1 - p_i) = \beta\lambda_i(1 - p_i)z'(u_i) \qquad (12)$$

for all i and

$$B = \sum_i \lambda_i(p_i z(U_i) + (1 - p_i)z(u_i)), \qquad (13)$$

where, to shorten notation, we define $B = \Sigma_i\lambda_i B_i(q_i)$.

These equations together imply,

$$p_i w'(q_i^*) = \gamma_i \qquad (14)$$

$$z'(U_i^*) = \frac{\alpha_i}{\beta\lambda_i} = z'(u_i^*), \qquad (15)$$

$$\frac{z'(U_1^*)}{z'(U_2^*)} = \frac{\alpha_1/\lambda_1}{\alpha_2/\lambda_2}. \qquad (16)$$

It follows that first-best optimality implies *full insurance*, that is, for all i,

$$U_i^* = u_i^*, \qquad (17)$$

and if $\alpha_i = \lambda_i$ for all i, in addition, we get full equality, *i.e.*, $U_1^* = U_2^*$. These results are standard consequences of risk aversion.

The above results describe an extremely idealized situation in which any degree of redistribution is possible, and politically acceptable. Note that full insurance implies $w(q_i^*) - R_i^* = w_0 - r_i^*$. Under Assumption 2, this implies $R_i^* - r_i^* = w(q_i^*) - w_0 > 0$, or $w(q_i^*) - R_i^* - w_0 =$

$-r_i^*$. So, if we require $w(q_i^*) - R_i^* \geq w_0$, *i.e.*, if we want individuals to receive weakly more in case of success than if they were not educated, this implies $r_i^* \leq 0$.

Note that there doesn't exist an unconstrained optimum with $r_i^* \geq 0$ for all i. If such an optimum did exist, then, because of full insurance, we would have $w(q_i^*) - R_i^* = w_0 - r_i^* \leq w_0$ or, equivalently, $U_i^* \leq u(w_0)$. But we would also have $u_i^* \leq u(w_0)$ and $\Sigma_i \lambda_i (p_i z(U_i^*) + (1 - p_i) z(u_i^*)) \leq w_0 < B^*$, a contradiction, since resources would then be wasted.

If we do not permit negative repayments (*i.e.*, if the banker is not an insurer), optimality implies $r_i^* = 0$: we find a *contingent reimbursement loan*, in the ordinary sense that no repayment is required in case of "failure".

The logic of political acceptability of the loan and transfer schemes should also lead to consideration of individual rationality constraints for each type. We take these constraints to be *interim* participation constraints, that is, for all i,

$$p_i u(w(q_i) - R_i) + (1 - p_i) u(w_0 - r_i) \geq u_0 = u(w_0). \tag{IR$_i$}$$

IR_i means that type i prefers to participate in the loan scheme with education over getting the basic wage w_0 for sure. We will also restrict the discussion of Pareto optima to allocations satisfying IR$_i$ for all i. Note that if the solution satisfies IR_i, by concavity of u, we have $u[p_i(w(q_i^*) - R_i^*) + (1 - p_i)(w_0 - r_i^*)] \geq u(w_0)$ and therefore

$$p_i(w(q_i^*) - R_i^*) + (1 - p_i)(w_0 - r_i^*) \geq w_0. \tag{18}$$

Now, reintroducing cross-subsidies, optimality also requires $p_i R_i^* + (1 - p_i) r_i^* = \gamma_i q_i^* + t_i^*$. Substituting these relations in the above inequality yields $B_i(q_i^*) - w_0 > t_i^*$. This puts an upper bound on the exploitation of type i by means of taxes. The IR constraints imply that type i's higher-education surplus cannot be fully extracted by the transfer system. The IR constraints will typically hold strictly if the ratios α_i / λ_i are not too different from 1.

If we now combine IR_i with $r_i^* = 0$, then we obtain $p_i u(w(q_i^*) - R_i^*) \geq p_i u_0$ or equivalently, $w(q_i^*) - R_i^* \geq w_0$. Conversely, $w(q_i^*) - R_i^* \geq w_0$ and $r_i^* = 0$ imply IR_i. We conclude that if $r_i = 0$, IR_i holds if and only if $R_i \leq w(q_i^*) - w_0$.

2.3 First-best optimality when negative repayments are not permitted

The first-best optimality problem under the nonnegative repayment constraints can be formulated as follows. Maximize $\Sigma_i \alpha_i (p_i U_i + (1-p_i)u_i)$ with respect to $(q_i, U_i, u_i)_{i=1,2}$, subject to the resource constraint $\overline{RC}$, and the nonnegative repayment constraints

$$u_i \leq u_0, \qquad (\underline{NR}_i)$$

and

$$U_i \leq \overline{u}_i, \qquad (\overline{NR}_i)$$

where, by definition, $u_0 = u(w_0)$ and $\overline{u}_i = u(w(q_i))$. If we assume that the solution satisfies $U_i < \overline{u}_i$, introducing the multipliers δ_i for the constraints $u_i \leq u_0$, $i = 1, 2$, and β for the resource contraint $\overline{RC}$, we easily get the following necessary conditions (*i.e.*, Kuhn-Tucker conditions). The problem being convex, these conditions are also sufficient.

$$\begin{aligned}
\frac{z'(U_1^*)}{z'(U_2^*)} &= \frac{\alpha_1/\lambda_1}{\alpha_2/\lambda_2}, \\
0 &= \alpha_i - \lambda_i \beta z'(U_i^*), \\
\delta_i &= (1-p_i)[\alpha_i - \lambda_i \beta z'(u_i^*)],
\end{aligned} \qquad (19)$$

$$\beta \geq 0, \quad \delta_i \geq 0, \quad \text{and} \quad \delta_i(u_i^* - u_0) = 0.$$

The resource constraint, holding as an equality, should be added to the above list. The solution is the same as before, except that the full insurance property does not hold: we have $u_i^* = u_0$ instead. We must also have $U_i^* > u_0$, for otherwise, available resources would not be exhausted (and IR constraints would be binding). Remark now that since $z'(.) > 0$, we must have $\beta > 0$, and

$$\beta z'(u_0) < \beta z'(U_i^*) = \frac{\alpha_i}{\lambda_i}.$$

We must check that all multipliers, and therefore δ_i^*, are nonnegative. Using (19), we get $\delta_i^* > 0$. This confirms that $u_i^* = u_0$ is the right solution here.

Using the resource constraint and the relations $z(u_0) = w_0$ and $z(U_i^*) = w(q_i^*) - R_i^*$, it is easy to check that $\overline{RC}$ boils down to

$$\sum_i \lambda_i \gamma_i q_i^* = \sum_i \lambda_i p_i R_i^*, \qquad (20)$$

that is, total education costs are covered by the total sum of expected reimbursements[10].

Again, we get full equality (without full insurance) in the particular case in which $\alpha_i = \lambda_i$, since in this case, the first necessary condition above implies $U_1^* = U_2^*$. Full equality is driven by risk aversion, since in this utilitarian framework, risk aversion determines a certain degree of social aversion for inequality. A benevolent utilitarian planner would like to insure each type of agent against the consequences of failure, but would also insure every individual, *ex ante*, against the risk of being less talented: hence the equalization property.

Since there are no repayments in case of failure, the resource constraint imposes $R_i^* > 0$ for at least one i. We neglect possible corner solutions in which $R_i^* = 0$ for some i. These corner solutions may appear[11] for values of α_1 that are small enough (or large enough). So, we in fact assume that in an open neighborhood of λ_1, the values of α_1 generate solutions that are interior in the sense that $R_i^* > 0$ for all i.

3 Asymmetric Information and Second-Best Optima

Let us now study the case in which types are not observed by public authorities, the banker and the higher education institutions. By definition, second-best optimal (or *interim efficient*) allocations maximize a weighted sum of the student's expected utilities, subject to resource-feasibility and incentive-compatibility (hereafter, the IC constraints). Students self-select in a menu of contracts proposed by the public authorities. The allocation determines utility values (U_i, u_i) a quality q_i and a subsidy t_i for each type i.

[10] Note that $u_i^* = u_0$ and $U_i^* = \overline{u}_i$ for all i is impossible since this would imply $\Sigma_i \lambda_i (p_i z(U_i^*) + (1 - p_i) z(u_i^*)) = \Sigma_i \lambda_i (p_i S(q_i^*) + (1 - p_i) w_0) > B^*$, and as soon as $q_i^* > 0$ (something we assume here), the latter inequality shows that the resource constraint would then be violated.

[11] There are also corner solutions in which either IR_1 or IR_2 are binding, when the welfare weights α_i take extreme values. We do not decribe these solutions in details here. Note first that IR constraints cannot bind simultaneously (this would contradict the resource-exhaustion property). If α_2 is sufficiently larger than λ_2, (or equivalently, α_1 is sufficiently smaller than λ_1) we find an optimal first-best solution in which IR_1 is binding (type 1 is fully exploited). We then obtain $u_1^* = U_1^* = u_0 = u_2^*$ and U_2^* is determined by the resource constraint. The same type of corner solution holds, mutatis mutandis, if α_2 is sufficiently smaller than λ_2, and IR_2 is then binding.

3.1 Incentive compatibility constraints

An allocation is incentive compatible if it satisfies the following constraint, for all i and all $j \neq i$, $i, j = 1, 2$.

$$p_i U_i + (1 - p_i) u_i \geq p_i U_j + (1 - p_i) u_j, \tag{IC_i}$$

This formulation is standard. Adding up the IC constraints immediately yields

$$(p_2 - p_1)(U_2 - U_1) \geq (p_2 - p_1)(u_2 - u_1)$$

and $p_2 > p_1$ implies the property

$$U_2 - u_2 \geq U_1 - u_1. \tag{D}$$

This property has important consequences. If type 1 is fully insured, *i.e.*, $U_1 = u_1$, then type 2 gets more in the good state, *i.e.*, $U_2 \geq u_2$. But if type 2 is fully insured, *i.e.*, $U_2 = u_2$, then, type 1 gets more in the bad state, *i.e.*, $u_1 \geq U_1$.

Another consequence of the IC constraints is the following. Since IC_i can be rewritten $p_i(U_i - U_j) \geq (1 - p_i)(u_j - u_i)$, we get the string of inequalities,

$$\frac{p_2}{1 - p_2}(U_2 - U_1) \geq u_1 - u_2 \geq \frac{p_1}{1 - p_1}(U_2 - U_1). \tag{IC}$$

An immediate consequence is the following.

Result 1.
IC constraints imply

$$U_2 \geq U_1, \tag{21}$$

and,

$$u_1 \geq u_2. \tag{22}$$

Proof: Since $p_2 > p_1$, if U_1 was strictly greater than U_2 we would get a contradiction. IC above shows that $U_2 \geq U_1$ implies $u_1 - u_2 \geq 0$.

Q.E.D.

Result 2.

a) If IC_1 and IC_2 are simultaneously binding, then $U_2 = U_1$ and $u_1 = u_2$: we get *equal treatment* (but not necessarily full insurance).

b) If equal treatment doesn't hold, then, either IC_1 or IC_2 is binding or none of them (but not both).

c) Under IC_1 and IC_2, then $u_1 = u_2$ if and only if $U_2 = U_1$.

Proof: The proofs of Results 2a and 2b are trivial, since $p_2 > p_1$. Result 2c follows from the fact that $u_1 = u_2$ and IC imply $U_2 - U_1 \geq 0 \geq U_2 - U_1$ and therefore $U_2 = U_1$. But we also have that IC and $U_2 = U_1$ imply $u_1 = u_2$.
Q.E.D.

3.2 Second-best optimality when cross-subsidies are not allowed

We start with the simplest case in which cross-subsidies between types are not allowed. This case is closest to a model of discrimination à la Rothschild-Stiglitz in which perfect competition between lenders would forbid cross-subsidies[12]. In this special case, there are two resource constraints (or two profitability constraints). For each type i, we must have $p_i R_i + (1 - p_i) r_i \geq \gamma_i q_i$. Using the inverse utility formulation, the latter constraint is equivalent to

$$p_i(w(q_i) - z(U_i)) + (1 - p_i)(w_0 - z(u_i)) \geq \gamma_i q_i,$$

or equivalently,

$$B_i(q_i) \geq p_i z(U_i) + (1 - p_i) z(u_i). \tag{RC_i}$$

The second-best optimum problem (or *interim incentive efficiency* problem) can be stated as follows.

Maximize $\Sigma_i \alpha_i (p_i U_i + (1 - p_i) u_i)$, with respect to $(q_i, U_i, u_i)_{i=1,2}$, subject to the constraints IC_i, and RC_i, for all i.

We assume that the solution satisfies IR_i constraints as strict inequalities (which is the typical case here). Note that λ doesn't play any role here, since there are no transfers of

[12] See Rothschild and Stiglitz (1976), Wilson (1977).

income between types. Note in addition that we ignore the nonnegative repayment constraint too.

The first-order (*i.e.*, Kuhn-Tucker) conditions of the second-best optimality problem are the following:

$$\begin{aligned}
\beta_i B_i'(q_i) &= 0 & \text{(FOC0)}\\
\alpha_1 p_1 + \mu_1 p_1 - \mu_2 p_2 &= \beta_1 p_1 z'(U_1); & \text{(FOC1)}\\
\alpha_2 p_2 + \mu_2 p_2 - \mu_1 p_1 &= \beta_2 p_2 z'(U_2); & \text{(FOC2)}\\
\alpha_1(1-p_1) + \mu_1(1-p_1) - \mu_2(1-p_2) &= \beta_1(1-p_1)z'(u_1); & \text{(FOC3)}\\
\alpha_2(1-p_2) + \mu_2(1-p_2) - \mu_1(1-p_1) &= \beta_2(1-p_2)z'(u_2); & \text{(FOC4)}
\end{aligned}$$

where $\beta_i \geq 0$ is the Lagrange multiplier of RC_i and μ_i is the multiplier of IC_i. To these conditions we must add the original RC and IC constraints and the complementary slackness equalities. Note that in the above conditions, all multipliers must be nonnegative. Using familiar techniques, it is possible to prove directly that we must have $q_i = q_i^*$ in this second-best problem[13]. The first result that we state shows which IC constraint is binding at the optimum.

Lemma 1. If cross-subsidies are not permitted, and if nonnegative repayment constraints are ignored, at a second-best optimum, IC_1 is binding.

For proof, see the Appendix.

Using Lemma 1, we can now prove the following proposition.

Proposition 1. If cross-subsidies are not permitted, and if nonnegative repayment constraints are ignored, at a second-best optimum, IC_1 is binding and IC_2 is slack. The second-best optimal solution is such that education levels are first-best efficient, *i.e.*, $q_i = q_i^*$; RC_i is binding for all i; we have,

$$U_2^{**} > U_1^{**} = u_1^{**} > u_2^{**},$$

[13] An alternative route is to prove that $\beta_i > 0$, since $B_i'(q_i^*) = 0$ (for instance, see the Appendix, this is a by-product of the proof of Proposition 1).

where $U_1^{**} = u(B_1(q_1^*))$, and (U_2^{**}, u_2^{**}) is determined by the intersection of IC_1 and RC_2.

For proof, see the Appendix.

Note in passing that the second-best allocation of Proposition 1 is independent of α (but Lagrange multipliers are functions of α). We state this result as a corollary.

Corollary 1. The second-best optimal solution of Proposition 1, obtained when cross-subsidies are not permitted, and when the nonnegative repayment constraints are ignored, is independent of α, that is,

$$\frac{\partial}{\partial \alpha_2}(U_2^{**}, u_2^{**}, U_1^{**}, u_1^{**}) = 0.$$

Proof: The proof of Proposition 1 shows that the menu of contracts $((U_1^{**}, u_1^{**}), (U_2^{**}, u_2^{**}))$ is fully determined by four equations: $U_1^{**} = u_1^{**}$ (first equation); RC_1 fully pins down (U_1^{**}, u_1^{**}) (second equation); while (U_2^{**}, u_2^{**}) is determined, knowing U_1^{**}, by the intersection of RC_2 (third equation) and IC_1 (fourth equation). None of these equations involves parameter α. *Q.E.D.*

This second-best optimum can be decentralized as a competitive equilibrium in the sense of Rothschild-Stiglitz (1976), with the menu of contracts $((U_1^{**}, u_1^{**}), (U_2^{**}, u_2^{**}))$, under the usual conditions. To check this, let $((\widehat{u}, \widehat{U}), (\widehat{u}, \widehat{U}))$ denote a *pooling* contract. Strictly speaking, this contract pools types in the utility space, but it does not necessarily pool types in the contract space. Pooling in the usual sense, *i.e.*, in the contract space, means $q_1 = q_2$, $r_1 = r_2$, $R_1 = R_2$; pooling in the utility space does not imply pooling in the contract space since $U_1 = U_2$ implies $w(q_2^*) - w(q_1^*) = R_2 - R_1$, and therefore $R_2 > R_1$. By definition, the pooling allocation $(\widehat{u}, \widehat{U})$ is feasible if and only if,

$$p_\lambda z(\widehat{U}) + (1 - p_\lambda) z(\widehat{u}) \leq \sum_i \lambda_i B_i(q_i^*) = B^*,$$

where by definition, $p_\lambda = \lambda_1 p_1 + \lambda_2 p_2$. This contract has the desirable full-insurance property if and only if $\widehat{u} = \widehat{U}$, and feasibility then implies $z(\widehat{u}) = \sum_i \lambda_i B_i(q_i^*)$ or $\widehat{u} = u(B^*)$. By definition, $u(B_1^*) < \widehat{u}$, so that type 1 prefers the pooling contract to (u_1^{**}, u_1^{**}). Type 2 is not attracted by this contract if and only if,

$$p_2 U_2^{**} + (1 - p_2) u_2^{**} > \widehat{u} = u(B^*),$$

or, equivalently,

$$z[p_2 U_2^{**} + (1 - p_2) u_2^{**}] > B^*. \tag{NP}$$

But the convexity of z and RC_2 imply,

$$z[p_2 U_2^{**} + (1 - p_2) u_2^{**}] < p_2 z(U_2^{**}) + (1 - p_2) z(u_2^{**}) = B_2(q_2^*).$$

Clearly, the *no-pooling* inequality (*i.e.*, NP) will be violated if λ_1, the frequency of type 1, is low enough, for in this case, B^* would be very close to $B_2(q_2^*)$. We can therefore state the following result.

Proposition 2. If the frequency λ_1 of type 1 is high enough, so that NP holds, the second-best optimum, $((U_1^{**}, u_1^{**}), (U_2^{**}, u_2^{**}))$, where $U_1^{**} = u_1^{**} = u(B_1(q_1^*))$, and where (U_2^{**}, u_2^{**}) is determined by the intersection of IC_1 and RC_2, is a competitive equilibrium of the lending market in the sense of Rothschild-Stiglitz.

3.3 Second-best optimality with cross-subsidies

We now study the more complicated case in which cross-subsidies are permitted, and the nonnegative repayment constraints are imposed. To fix ideas, assume that a public lending agency chooses the menu of loans proposed to students and implements the redistributive taxes and subsidies. The public agency should try to maximize $\Sigma_i \alpha_i (p_i U_i + (1 - p_i) u_i)$ with respect to (U_i, u_i), $i = 1, 2$ subject to IC_i, $\overline{RC}$ and $\underline{NR}_i$, $i = 1, 2$. We assume that the solution satisfies the IR constraints and $\overline{NR}$ constraints to clarify the analysis. We expect that the latter constraints will typically be satisfied if the choice of weights α is not too extreme. In addition, IC and $\underline{NR}_1$ together imply $\underline{NR}_2$, since we must have $u_2 \leq u_1 \leq u_0$. It follows that we can ignore $\underline{NR}_2$. Let then β, μ_i, $i = 1, 2$ and δ be the Lagrange multipliers of $\overline{RC}$, IC_i, and $\underline{NR}_1$ respectively. Kuhn and Tucker's Theorem tells us that these multipliers must be nonnegative. The first-order conditions for a second-best optimum (or *interim efficient allocation*) can easily be derived, as follows,

$$\begin{aligned}
\beta\lambda_i B_i'(q_i) &= 0 && \text{(FOC0)}\\
\alpha_1 p_1 + \mu_1 p_1 - \mu_2 p_2 &= \beta\lambda_1 p_1 z'(U_1); && \text{(FOC1)}\\
\alpha_2 p_2 + \mu_2 p_2 - \mu_1 p_1 &= \beta\lambda_2 p_2 z'(U_2); && \text{(FOC2)}\\
\alpha_1(1-p_1) + \mu_1(1-p_1) - \mu_2(1-p_2) &= \beta\lambda_1(1-p_1)z'(u_1) + \delta; && \text{(FOC3)}\\
\alpha_2(1-p_2) + \mu_2(1-p_2) - \mu_1(1-p_1) &= \beta\lambda_2(1-p_2)z'(u_2). && \text{(FOC4)}
\end{aligned}$$

And we must add the complementary slackness conditions, that is, $\delta(u_1 - u_0) = 0$, $\beta[B^* - \Sigma_i\lambda_i(p_i z(U_i) + (1-p_i)z(u_i))] = 0$, etc. From these conditions, we can derive a number of results. Note first that, adding FOC1 and FOC2, we easily derive

$$\beta = \frac{\alpha_1 p_1 + \alpha_2 p_2}{\Sigma_i \lambda_i p_i z'(U_i)} > 0.$$

Therefore $\overline{RC}$ is binding and from FOC0, education levels are again undistorted, *i.e.*, $B_i'(q_i) = 0$ and $q_i = q_i^*$ for all i. We then easily derive the following crucial Lemma.

Result 3. At a second-best optimal solution, if IC_2 is binding then IC_1 is also binding.

Proof. If IC_2 is binding and IC_1 is slack, we have $\mu_1 = 0$. Since IC and IR_1 constraints hold, we know from Result 1 that

$$U_2 > U_1 > u_0 \geq u_1 > u_2.$$

FOC2 and FOC4 yield,

$$\begin{aligned}
\alpha_2 + \mu_2 &= \beta\lambda_2 z'(U_2); && \text{(FOC4a)}\\
\alpha_2 + \mu_2 &= \beta\lambda_2 z'(u_2). && \text{(FOC2a)}
\end{aligned}$$

Since $\beta > 0$, it immediately follows that $z'(U_2) = z'(u_2)$, and therefore, $u_2 = U_2$. This is a contradiction since this implies $U_2 = U_1 = u_0 = u_1 = u_2$ under IC constraints. We conclude that both IC constraints must be binding if IC_2 is binding.
Q.E.D.

In this new context, the solution varies with the social weights α. The next result gives us one of the possible structures of the solution, with IC_1 binding. This solution arises if the talented are sufficiently overweighted, that is, if α_2 is sufficiently higher than λ_2.

Proposition 3. (*The talented bear more risk when favored*). Suppose that a second-best solution satisfies IR and $\overline{NR}$ constraints as strict inequalities, then, IC constraints are not both binding, *only if* $\alpha_2 > \lambda_2$ (overweighting of the talented), and then

$$U_2^{**} > U_1^{**} > u_0 = u_1^{**} > u_2^{**}. \tag{23}$$

The solution, if it exists, is determined by the intersection of $u_1^{**} = u_0$, IC_1, $\overline{RC}$ expressed as equalities and the necessary condition

$$\frac{\alpha_2[\Sigma_i \lambda_i p_i z'(U_i^{**})]}{\lambda_2[\alpha_1 p_1 + \alpha_2 p_2]} = \frac{[p_2(1-p_1)z'(U_2^{**}) - p_1(1-p_2)z'(u_2^{**})]}{[p_2 - p_1]}. \tag{C}$$

For proof, see the Appendix.

Remark that in the cases described by Proposition 3, that is, when α_2 is large enough, the second-best allocation depends on parameter α. Now, in contrast, if α_2 is close to λ_2, or if α_2 is smaller than λ_2, the second-best solution is *egalitarian* (in a certain sense) with *maximal insurance* (*i.e.*, we find an allocation with *contingent reimbursement loans,* not the same thing as full insurance), and involves a form of *bunching* with respect to α.

Proposition 4. (*Equal treatment as a second best*). Suppose that a second-best solution satisfies IR and $\overline{NR}$ constraints as strict inequalities, and that $u_1^{**} = u_2^{**} = u_0$. Then, we find a second-best optimum such that IC_1 and IC_2 are both binding, $\Sigma_i \lambda_i p_i R_i^{**} = \Sigma_i \lambda_i \gamma_i q_i^*$ and $U_1^{**} = U_2^{**}$, *if* α_2 is smaller than a threshold L_2, with $\lambda_2 < L_2 < 1$.
For proof, see the Appendix.

Again we can state a corollary on the local effect of α on the second-best allocation.

Corollary 2. (*Bunching with respect to* α) The second-best optimal solution of Proposition 4, obtained when cross-subsidies are permitted, but when nonnegative repayment constraints are imposed, is independent of α, when $\alpha_2 < L_2$, we have,

$$\frac{\partial}{\partial \alpha_2}(U_2^{**}, u_2^{**}, U_1^{**}, u_1^{**}) = 0.$$

Proof: From the proof of Proposition 4, we see that the menu of contracts $((U_1^{**}, u_1^{**}),$ $(U_2^{**}, u_2^{**}))$ is fully determined by four equations: *(i)*, $u_1^{**} = u_0$; *(ii)*, $u_2^{**} = u_0$; *(iii)*, *IC* constraints then imply $U_1^{**} = U_2^{**} = U^{**}$; *(iv)*, given the values of (u_1^{**}, u_2^{**}), the remaining value of U^{**} is pinned down by $\overline{RC}$. None of these equations involves parameter α.
Q.E.D.

From Propositions 3 and 4 and from the fact that the FOCs are necessary and sufficient in our model, we conclude that the second-best optimum's structure depends on the social welfare weight α_2 in a simple way.

Corollary 3. (*Optimality of contingent reimbursement loans*) Among the second-best optima satisfying IR and $\overline{NR}$ constraints as strict inequalities, there are two types of allocations. First, if the weight of talented types is not too high, more precisely, if $\alpha_2 \leq L_2$ where L_2 is a threshold such that $\lambda_2 < L_2 < 1$, the second-best optimum equalizes opportunities, in the particular, but legitimate sense that $u_1^{**} = u_2^{**}$ and $U_1^{**} = U_2^{**}$: we call this the *equal treatment property.* In addition, the second-best provides maximal insurance under nonnegative repayment constraints (*i.e.*, in the bad state, there are no repayments). If $\alpha_2 > L_2$, the second-best optimum is such that $U_2^{**} > U_1^{**} > u_0 = u_1^{**} > u_2^{**}$, that is, maximal insurance is provided to the less talented in the form of a contingent repayment loan, with zero repayment in the bad state, while, due to IC constraints, the talented types are less than maximally insured: they face higher risks in the form of positive repayments in the bad state and higher rewards in the good state.

Remark 1. (*Equal treatment or pooling?*) Note that the efficient allocations of Proposition 4, are not *pooling contracts* in the usual sense. This is because, according to usual terminology, a pooling contract should be defined by the property $r_1^{**} = r_2^{**}$, $R_1^{**} = R_2^{**}$ *and* $q_1^{**} = q_2^{**}$. Here, we have $r_1^{**} = r_2^{**} = 0$ but $U_1^{**} = U_2^{**}$ implies $R_2^{**} > R_1^{**}$ since we have $q_1^{**} = q_1^{*} < q_2^{**} = q_2^{*}$, and $R_2^{**} - R_1^{**} = w(q_2^{*}) - w(q_1^{*}) > 0$. Hence, the talented reimburse more than the less-talented, in case of success. The "egalitarian" solution of Proposition 4 would be a pooling contract in the usual sense, *i.e.*, as in a simple Rotschild-Stiglitz model of insurance under adverse selection if we imposed $q_1 = q_2$ or if $w(q)$ was a constant. But the different student-types choose different levels of education and therefore, the two IC constraints can be simultaneously binding while the repayment profile of the types is not the same. In other words, pooling takes place in the net income space, but not in the contract space[14].

Remark 2. (*Generic cross-subsidization*) The incentive efficient allocations obtained with $\alpha_2 \leq \lambda_2$ may typically involve a cross-subsidy from the talented in favor of the less-talented. There is a cross-subsidy between types if $p_2 R_2^{**} \neq \gamma_2 q_2^*$, or equivalently, if $B_2^* \neq p_2 z(U_2^{**}) + (1-p_2) z(u_0)$. In essence, when $\alpha_2 \leq \lambda_2$, optimality with cross-subsidies involves equalization with $U_2^{**} = U_1^{**}$ and $u_2^{**} = u_1^{**} = u_0$, but, since $B_2^* > B_1^*$, given the resource constraint, the talented will typically subsidize the less talented. When $\alpha_2 > \lambda_2$, the second-best solution will typically also exhibit cross-subsidization. This is because efficiency requires some degree of insurance of the students against the ex ante risk of being *less* talented (and therefore, the risk of receiving a smaller income, even in case of success). This type of insurance motive exists for all values of α_2 but of course, redistribution disappears in the extreme case $\alpha_2 = 1$ (in which IR_1 is binding).

Remark 3. (*First-best and second-best allocations are different*) Note that, under nonegative repayment constraints, *the second-best doesn't coincide with the first best.* This is because first-best optimality involves $U_1^* = U_2^*$ only if $\alpha_2 = \lambda_2$ (see subsection 2.3. above). For values of α_2 strictly smaller than λ_2, full equality is not required by first-best optimality. Indeed, we have checked that the multipliers of IC constraints μ_1^{**} and μ_2^{**} are strictly positive, therefore, the first-best condition, *i.e.*, $\beta \lambda_i z'(U_i^*) = \alpha_i$ for all i is violated, as can be checked with a glance at FOC1 and FOC2. The equal treatment property is imposed as a way of taking care of incentive compatibility constraints, and thus, when $\alpha_2 \neq \lambda_2$, it is a source of sub-optimality.

Corollary 4. (*Generic suboptimality of second-best allocations*)
The second-best and first-best allocations coincide only when $\alpha_2 = \lambda_2$. The second-best allocation is however efficient when $\alpha_2 < L_2$, since $U_2^{**} = U_1^{**}$ and $u_2^{**} = u_1^{**} = u_0$ is one of the first-best optima, obtained when $\alpha_2 = \lambda_2$, *under nonegative repayment constraints*

Efficiency doesn't mean optimality here: this is because the choice of α picks a particular efficient allocation on the Pareto frontier. Corollary 4 says that the second best may be first-

[14] As usual if IC_1 and IC_2 are both binding, we assume that type i chooses $q_i = q_i^*$, $i = 1, 2$.

best efficient for $\alpha_2 < L_2$, because the second-best allocation is on the Pareto frontier, but it doesn't pick the right optimum, given α. This is obvious because the appropriate optimum is not egalitarian when $\alpha_2 \neq \lambda_2$. We have found a case of "equalization as a second-best": redistribution is not driven by the social planner's objective function or the welfare weights, it is just the best way of taking care of incentives for a range of welfare weights.

Remark 4. Equal treatment in the above sense is neither the ex post *nor the* ex ante *equality of types.* The equality result of Proposition 4 is not an equality of outcomes, neither between, nor within types. An independent, random chance draw determines the *ex post* result for each student, so, there will be successful and unsuccessful persons in the population of each type. *Ex post*, under the second-best optimal policy, all "losers" will be equal and all "winners" will be equal in *utility or net income terms*, whichever their type. But the types do not face the same repayment in case of success. Of course, there are more unsuccessful people in the population of type-1 students, since $p_1 < p_2$. It follows from this that, *ex ante*, type-2 students have a higher expected utility than type-1 students. It is in this sense that the second-best solution doesn't implement a full equality of outcomes.

Remark 5. (*Decentralization and cream-skimming*) It is not possible to find a simple equivalent of the *second welfare theorem* in this context. When the weight of talented types is close to their frequency in the population, or when $\alpha_2 < L_2$, the incentive efficient allocation exhibits equalization (or pooling in the utility space). If supplied by commercial banks under competition, the second-best optimal *menu of contrats* would just break even and yield zero profits. The cross-subsidization of high risks by low risks, which is typical of this situation, opens the possibility of competitive *cream-skimming* by an entering company. Suppose that a banker offers exactly the second-best optimal menu of contracts obtained when $\alpha_2 = 1$. This allocation is separating, it is also the best possible for the low-risk, talented types and necessarily, the individual rationality constraint IR_1 of the less-talented and IC_1 are binding. In addition, it is not difficult to see that this allocation entails maximal insurance for the less-talented, because it is the cheapest way of ensuring their participation. If such a menu is offered, *(i)*, the talented will obviously desert the pooling allocation, because they are exploited in favor of the less talented, and choose the new contract; *(ii)*, the old pooling menu of contracts will be rendered unprofitable and will be withdrawn from the market;

finally *(iii)*, the less-talented will choose the maximal insurance contract in the new menu, and the entrant breaks even. Competition will then favor the talented types, as already noted by Miyazaki[15] (1977). The competitive equilibrium in this market is called a Wilson-Miyazaki equilibrium (cf. Wilson (1977), Crocker and Snow (1985)). An equilibrium in the sense of Wilson-Miyazaki is a profitable menu of contracts such that no other menu can enter the market, attract customers, and earn a nonnegative profit, even after the menus rendered unprofitable by the new entry have been withdrawn[16]. The separating allocation which maximizes the utility of the talented under IC, IR, RC and nonnegative repayment constraints is a Wilson-Miyazaki equilibrium because any separating, second-best allocation which doesn't maximize the utility of the talented can be driven out of the market by the former menu. This means that, in general, incentive efficient allocations with cross-subsidies cannot be decentralized as market equilibria, even as Wilson-Miyazaki equilibria. Public intervention is needed to implement the second-best optimum with equal opportunities.

4 Moral Hazard and Adverse Selection Combined

Until now, we have studied a pure adverse selection problem in which individual effort doesn't play a rôle. But of course, moral hazard, that is, problems posed to lenders by the unobservable actions of borrowers, should be introduced into the picture. We endeavor to do just this with a simple representation of the student's hidden effort. The effort variable of type i, denoted e_i, can take two values only: 0 or 1. The cost of effort is type-dependent and defined as $c_i e_i$ where $c_i > 0$ is a parameter; it is additively separable. Effort e_i is strictly individual in the sense that it affects the probability of a good outcome for the individual who exerted it, without any influence on the outcomes of other individuals. The probability of success is a function of effort, *i.e.*, $p = p_i(e_i)$. To simplify notation, denote $P_i = p_i(1)$ and

[15] Miyazaki (1977) was one of the first to study competition by means of menus of contracts with cross-subsidies. See also Spence (1978).

[16] For a discussion of this equilibrium concept, and variants, see Hellwig (1987), Henriet and Rochet (1990), Dionne *et al.* (2000).

$p_i = p_i(0)$.

Assumption 3. We assume that effort raises the probability of success for each type, that is $P_i > p_i$, $i = 1, 2$, and $P_2 > P_1$; $p_2 > p_1$.

We now have a *generalized Principal-Agent problem* in the sense of Myerson (1982) and we apply the *extended revelation principle* (see also Laffont and Martimort (2002)). The constraints are now *revelation* and *obedience* constraints: the students should simultaneously self-select by choosing the right contract in the menu *and* exert the right amount of effort. Assuming that high effort is efficient (the only interesting case here), we can now write the incentive constraints as follows:

$$P_iU_i + (1 - P_i)u_i \geq P_iU_j + (1 - P_i)u_j, \quad (\overline{IC}_i)$$

$$P_iU_i + (1 - P_i)u_i - c_i \geq p_iU_i + (1 - p_i)u_i, \quad (MH_i)$$

$$P_iU_i + (1 - P_i)u_i - c_i \geq p_iU_j + (1 - p_i)u_j, \quad (\underline{IC}_i)$$

for all $i = 1, 2$ and $j \neq i$. Constraint $\overline{IC}_i$ says that type i should not be tempted to pose as type j while exerting high effort. Constraint MH_i says that type i should prefer to exert high effort over low effort and honestly revealing her (his) type. Constraint $\underline{IC}_i$ says that type i prefers high effort to low effort and posing as type j. Constraints $\overline{IC}_i$ being essentially the same as before, Results 1 and 2, properties D and IC still hold here, with P_i instead of p_i. It is not difficult to see that MH_i can be rewritten as, $(P_i - p_i)(U_i - u_i) \geq c_i$, or

$$U_i - u_i \geq K_i \quad (MH_i)$$

where by definition,

$$K_i = \frac{c_i}{P_i - p_i}. \quad (24)$$

Moral hazard will thus force a minimal gap between the reward of success and that of failure. It is natural to assume that type 2 is more efficient than type 1 while exerting effort. Formally, we assume the following.

Assumption 4. $K_1 \geq K_2 \geq 0$.

We then easily find the following results.

Result 4. Under Assumption 4, if $\overline{IC}_1$, $\overline{IC}_2$ and MH_1 hold, then MH_2 is satisfied.

Proof: From $\overline{IC}_i$, $i = 1, 2$, we derive condition D and we obtain the following string of inequalities:

$$U_2 - u_2 \geq U_1 - u_1 \geq K_1 \geq K_2,$$

so MH_2 holds true.

Q.E.D.

The $\underline{IC}_i$ constraints are an added difficulty, but we can in fact ignore them, as shown by Result 5.

Result 5. Under Assumption 4,

a) if $\overline{IC}_i$, $i = 1, 2$ and MH_1 hold, then $\underline{IC}_1$ is satisfied.

b) if $\overline{IC}_2$ is satisfied, and if, in addition, $\overline{IC}_1$ and MH_1 are binding, then, $\underline{IC}_2$ is satisfied.

Proof: *(a)* If $\overline{IC}_1$ holds, then,

$$(1 - P_1)(u_1 - u_2) \geq P_1(U_2 - U_1),$$

and since under IC, $U_2 - U_1 \geq 0$, and we assumed $P_1 > p_1$, we also have $(1 - p_1)(u_1 - u_2) \geq p_1(U_2 - U_1)$. But MH_1 implies $c_1 - (P_1 - p_1)(U_1 - u_1) \leq 0$. This trivially implies

$$(1 - p_1)(u_1 - u_2) \geq p_1(U_2 - U_1) + c_1 - (P_1 - p_1)(U_1 - u_1), \tag{25}$$

and rearranging terms we get the equivalent inequality,

$$P_1U_1 + (1 - P_1)u_1 - c_1 \geq p_1U_2 + (1 - p_1)u_2,$$

but this is exactly $\underline{IC}_1$.

(b) Given that MH_1 is binding, $\underline{IC}_2$ can be expressed as follows,

$$P_2U_2 + (1 - P_2)u_2 - c_2 \geq p_2(u_1 + K_1) + (1 - p_2)u_1 = u_1 + p_2K_1. \tag{$\underline{IC}_2 + MH_1$}$$

Combining $\overline{IC}_1$ and MH_1, holding as equalities, we easily obtain,

$$u_1 + P_1K_1 = P_1U_2 + (1 - P_1)u_2. \tag{$\overline{IC}_1 + MH_1$}$$

Substituting the value of u_1 derived from $(\overline{IC}_1 + MH_1)$ in $(\underline{IC}_2 + MH_1)$ yields, after some rearrangement of terms,

$$(P_2 - P_1)(U_2 - u_2) \geq c_2 + (p_2 - P_1)K_1.$$

Dividing both sides by $(P_2 - p_2) > 0$ and rearranging terms, we obtain,

$$(U_2 - u_2)\left[\frac{P_2 - P_1}{P_2 - p_2}\right] \geq K_2 + \frac{p_2 - P_1}{P_2 - p_2}K_1.$$

From condition D and MH_1 we know that $U_2 - u_2 \geq U_1 - u_1 = K_1$. In addition, $(P_2 - P_1)/(P_2 - p_2) = 1 + (p_2 - P_1)/(P_2 - p_2) > 0$. Hence, the following string of inequalities:

$$(U_2 - u_2)\left[\frac{P_2 - P_1}{P_2 - p_2}\right] \geq K_1\left[1 + \frac{p_2 - P_1}{P_2 - p_2}\right] \geq K_2 + \frac{p_2 - P_1}{P_2 - p_2}K_1,$$

since, by Assumption 4, $K_1 \geq K_2$. This shows that $\underline{IC}_2$ is satisfied when $\overline{IC}_2$ holds and when $\overline{IC}_1$ and MH_1 are equalities.
Q.E.D.

The second-best optimality problem can now be defined. The benevolent public banker should maximize $\Sigma_i \alpha_i (P_i U_i + (1 - P_i)u_i - c_i)$ with respect to (U_i, u_i), $i = 1, 2$ subject to $\overline{RC}$, $\overline{IC}_i$, $i = 1, 2$, MH_1 and $\underline{IC}_2$. (We assume again that IR_i constraints are satisfied by the solution.) To study this problem, we will also temporarily ignore (*i.e.*, relax) constraint $\underline{IC}_2$ and check at the end the conditions under which it is satisfied. Let β, δ, μ_1 and μ_2 be the nonnegative Lagrange multipliers of, respectively, constraints $\overline{RC}$, MH_1, $\overline{IC}_1$ and $\overline{IC}_2$. The first-order conditions (*i.e.*, Kuhn-Tucker conditions) for the second-best optimality problem are the following.

$$\beta\lambda_i B_i'(q_i) = 0 \quad \text{(FOC0c)}$$

$$\alpha_1 P_1 + \mu_1 P_1 - \mu_2 P_2 + \delta = \beta\lambda_1 P_1 z'(U_1); \quad \text{(FOC1c)}$$

$$\alpha_2 P_2 + \mu_2 P_2 - \mu_1 P_1 = \beta\lambda_2 P_2 z'(U_2); \quad \text{(FOC2c)}$$

$$\alpha_1(1 - P_1) + \mu_1(1 - P_1) - \mu_2(1 - P_2) - \delta = \beta\lambda_1(1 - P_1)z'(u_1); \quad \text{(FOC3c)}$$

$$\alpha_2(1 - P_2) + \mu_2(1 - P_2) - \mu_1(1 - P_1) = \beta\lambda_2(1 - P_2)z'(u_2); \quad \text{(FOC4c)}$$

with the complementary slackness conditions, *i.e.*, $\delta(U_1 - u_1 - K_1) = 0$, $\beta[B^* - \Sigma_i\lambda_i(P_i z(U_i) + (1 - P_i)z(u_i))] = 0$, (in the expression of B^*, P_i should now be used instead of p_i), etc.

These conditions are necessary and sufficient for an optimum, because as noted above, the problem is convex. It follows from this that if we find a solution in which all multipliers are nonnegative, we have found the solution.

We now prove two useful preliminary results, the proof of which relies on first-order conditions.

Result 6. If $\overline{IC}_1$ and $\overline{IC}_2$ are binding, then, MH_1 must be binding at the second-best optimum.

Proof: If $\overline{IC}_1$ and $\overline{IC}_2$ are binding, then $U_1 = U_2 = U$ and $u_1 = u_2 = u$. Suppose that MH_1 is slack, *i.e.*, $U_1 > u_1 + K_1$, at the second-best optimum, then $\delta = 0$. Adding equations FOC1c to FOC4c, we easily find,

$$\beta = \frac{1}{P_\lambda z'(U) + (1 - P_\lambda)z'(u)} > 0,$$

where $P_\lambda = P_1\lambda_1 + P_2\lambda_2$. With $\delta = 0$, FOC1c and FOC3c form a linear system in (μ_1, μ_2); that is,

$$\begin{aligned} \mu_1 P_1 - \mu_2 P_2 &= P_1[\beta\lambda_1 z'(U) - \alpha_1]; \\ \mu_1(1 - P_1) - \mu_2(1 - P_2) &= (1 - P_1)[\beta\lambda_1 z'(u) - \alpha_1]; \end{aligned}$$

This system has a nonzero determinant, equal to $P_2 - P_1 > 0$, and a unique solution (μ_1^*, μ_2^*). It is easy to check that,

$$\mu_2^* = \frac{P_1(1 - P_1)}{P_2 - P_1}\beta\lambda_1\left(z'(u) - z'(U)\right).$$

But, now, MH_1 implies $U > u$, hence $z'(u) - z'(U) < 0$ and $\mu_2^* < 0$. This is a violation of Kuhn-Tucker conditions, since all multipliers must be non-negative. We have found a contradiction.

Q.E.D.

We then find that if a single IC constraint is binding at the optimum, it must be $\overline{IC}_1$.

Result 7. At the second-best optimum, if $\overline{IC}_2$ is binding, then, $\overline{IC}_1$ must be binding too.

Proof: If $\overline{IC}_2$ is binding, and $\overline{IC}_1$ is slack, then $\mu_1 = 0$. Adding the four FOC conditions together, we easily check that $\beta > 0$. Using FOC2c and FOC4c, we almost immediately obtain,

$$\beta\lambda_2 z'(u_2) = \alpha_2 + \mu_2 = \beta\lambda_2 z'(U_2),$$

and therefore, $u_2 = U_2$. This contradicts the fact that $\overline{IC}_1$ is slack.
Q.E.D.

A further study of FOCs yields the following proposition.

Proposition 5. (*Equal treatment as a second best under moral hazard and adverse selection*) Consider the second-best solutions satisfying IR constraints as strict inequalities. There exists a nonempty open interval $(\underline{L}_2, \overline{L}_2)$, including λ_2, such that if $\alpha_2 \in (\underline{L}_2, \overline{L}_2)$, then, under moral hazard and adverse selection, the second-best optimal solution has the following properties:

$$\widetilde{U}_1 = \widetilde{U}_2 = \widetilde{U}, \qquad \widetilde{u}_1 = \widetilde{u}_2 = \widetilde{u} \quad \text{(equal treatment)},$$

$$\widetilde{U} = \widetilde{u} + K_1 \quad \text{(incomplete insurance)},$$

$\overline{RC}$, MH_1, $\overline{IC}_1$ and $\overline{IC}_2$ are all binding. If, in addition, $K_1 > K_2$, then MH_2, $\underline{IC}_1$ and $\underline{IC}_2$ hold as strict inequalities.

Proof: The FOCs can be rewritten,

$$\mu_1 P_1 - \mu_2 P_2 + \delta = P_1[\beta\lambda_1 z'(U_1) - \alpha_1]; \qquad \text{(FOC1d)}$$

$$\mu_2 P_2 - \mu_1 P_1 = P_2[\beta\lambda_2 z'(U_2) - \alpha_2]; \qquad \text{(FOC2d)}$$

$$\mu_1(1-P_1) - \mu_2(1-P_2) - \delta = (1-P_1)[\beta\lambda_1 z'(u_1) - \alpha_1]; \qquad \text{(FOC3d)}$$

$$\mu_2(1-P_2) - \mu_1(1-P_1) = (1-P_2)[\beta\lambda_2 z'(u_2) - \alpha_2]. \qquad \text{(FOC4d)}$$

Our optimum candidate exhibits equal treatment, $\widetilde{U}_1 = \widetilde{U}_2 = \widetilde{U}$, $\widetilde{u}_1 = \widetilde{u}_2 = \widetilde{u}$, since both $\overline{IC}$ constraints are binding. By Result 6 above, MH_1 must be binding too. This imposes $\widetilde{U} = \widetilde{u} + K_1$. Adding the four FOCd equations easily yields,

$$\widetilde{\beta} = \frac{1}{P_\lambda z'(\widetilde{U}) + (1 - P_\lambda) z'(\widetilde{u})} > 0, \qquad (26)$$

where $P_\lambda = P_1\lambda_1 + P_2\lambda_2$. It follows that $\overline{RC}$ is binding and that $(\widetilde{U}, \widetilde{u})$ is fully determined by the intersection of $\overline{RC}$ and MH_1. We also find that FOC0c implies $q_i = q_i^*$, since $\widetilde{\beta}\lambda_i > 0$. Adding FOC1d and FOC2d, we derive an expression for δ. Adding FOC3d and FOC4d, we derive another expression for δ. The two expressions must be equal, hence,

$$\widetilde{\delta} = P_\lambda \widetilde{\beta} z'(\widetilde{U}) - P_\alpha = (1 - P_\alpha) - (1 - P_\lambda)\widetilde{\beta} z'(\widetilde{u}), \tag{27}$$

and since $\widetilde{\delta} \geq 0$, we must have,

$$P_\lambda \widetilde{\beta} z'(\widetilde{U}) \geq P_\alpha \quad \text{and} \quad (1 - P_\alpha) \geq (1 - P_\lambda)\widetilde{\beta} z'(\widetilde{u}). \tag{28}$$

Substituting the value of $\widetilde{\beta}$ obtained above, it is easy to check that these two inequalities are equivalent and that we must have,

$$\frac{P_\alpha}{P_\lambda} \leq \frac{z'(\widetilde{U})}{P_\lambda z'(\widetilde{U}) + (1 - P_\lambda) z'(\widetilde{u})}. \tag{29}$$

Now, since $\widetilde{U} > \widetilde{u}$, we see that the above inequality yields an upper bound for α_2. The latter parameter may be greater than λ_2, since the right-hand-side ratio in (29) is greater than one.

Finally, we must check that the associated multipliers $\widetilde{\mu}_i$ are nonnegative. FOC2d and FOC4d provide us with a linear system of equations for (μ_1, μ_2). The determinant of this system is $P_2 - P_1 > 0$, so that there is a unique solution $(\widetilde{\mu}_1, \widetilde{\mu}_2)$. We easily derive,

$$\widetilde{\mu}_1 = \frac{\lambda_2 P_2 (1 - P_2)\widetilde{\beta}}{P_2 - P_1}[z'(\widetilde{U}) - z'(\widetilde{u})] > 0. \tag{30}$$

$$\widetilde{\mu}_2 = \frac{1}{P_2 - P_1}\left[\widetilde{\beta}\lambda_2[P_2(1 - P_1)z'(\widetilde{U}) - P_1(1 - P_2)z'(\widetilde{u})] - \alpha_2[P_2 - P_1]\right]. \tag{31}$$

Note that if $\lambda_2 = \alpha_2$, then $\widetilde{\mu}_2 > 0$. To see this, remark that when $\lambda_2 = \alpha_2$, the conditions on $\widetilde{\delta}$ above imply $\widetilde{\beta} z'(\widetilde{U}) > 1$ and $-\widetilde{\beta} z'(\widetilde{u}) > -1$. We use these inequalities to show that 0 is a lower bound for $\widetilde{\mu}_2$, as follows,

$$\widetilde{\mu}_2 > \frac{\lambda_2}{P_2 - P_1}\left[[P_2(1 - P_1) - P_1(1 - P_2)] - [P_2 - P_1]\right] = 0.$$

By continuity, there exists an interval $[\underline{L}_2, \overline{L}_2]$ of values of α_2, including λ_2, such that all multipliers are nonnegative. By Result 5, we know that both $\underline{IC}_i$ constraints are satisfied. By Result 4, we know that MH_2 is also satisfied.

Q.E.D.

Note that in the statement of Proposition 5, we cannot let α_2 go to zero because we consider only optima such that IR_2 is satisfied as a strict inequality. In this context, we again find a bunching property.

Corollary 5. (*Bunching with respect to* α) The second-best optimal solution of Proposition 5, obtained when cross-subsidies are permitted, under moral hazard and adverse selection, is independent of α, when α_2 is small enough, *i.e.*, if $\alpha_2 < \overline{L}_2$, we have,

$$\frac{\partial}{\partial \alpha_2}(\widetilde{U}_2, \widetilde{u}_2, \widetilde{U}_1, \widetilde{u}_1) = 0.$$

Proof: The second-best allocation is the solution of a system of four equations with four unknowns: *(i)*, $\widetilde{U}_1 = \widetilde{U}_2$; *(ii)*, $\widetilde{u}_1 = \widetilde{u}_2$; *(iii)*, $\widetilde{U} = \widetilde{u} + K_1$; and *(iv)*, given these constraints, $\overline{RC}$ pins down $\widetilde{u}$. None of these equations involve α.
Q.E.D.

The remaining question is to find the second-best optimal solution when $\alpha_2 > \overline{L}_2$. We look for a second-best allocation in which a single IC constraint is binding. Then, by Result 7, we know that $\overline{IC}_1$ is the binding constraint, and this can happen *only if* $\alpha_2 > \lambda_2$.

Proposition 6. Suppose that a second-best solution satisfies IR constraints as strict inequalities. If this second-best optimum has only one binding IC constraint, then, $\overline{IC}_1$ is binding, $\overline{IC}_2$ is slack, MH_1 and $\overline{RC}$ are binding; we have $U_2 > U_1 > u_1 > u_2$ and necessarily, $\alpha_2 > \lambda_2$. The second-best solution is fully determined by the following 4 equations: $\overline{IC}_1$, MH_1 and $\overline{RC}$, expressed as equalities, and the condition,

$$\frac{\lambda_2}{\lambda_1 \alpha_2} \frac{[P_2(1-P_\alpha)z'(U_2) - P_\alpha(1-P_2)z'(u_2)]}{(P_2 - P_1)} = P_1 z'(U_1) + (1-P_1)z'(u_1) \qquad \text{(F)}$$

Proof: If a second-best optimum has only one binding IC constraint, then, by Result 7, $\overline{IC}_1$ must be binding and $\overline{IC}_2$ is slack. Hence, $\widetilde{\mu}_2 = 0$. By Results 4 and 5, we can neglect MH_2 and $\underline{IC}_i$ constraints. Adding the four FOCc equations, we easily obtain,

$$1/\widetilde{\beta} = \Sigma_i \lambda_i (P_i z'(\widetilde{U}_i) + (1-P_i)z'(\widetilde{u}_i)),$$

and therefore, $\widetilde{\beta} > 0$. Hence, $\overline{RC}$ is binding. Suppose now that MH_1 is slack. Then, $\delta = 0$. From FOC1c and FOC3c, we easily derive,

$$\widetilde{\beta}\lambda_1 z'(\widetilde{u}_1) = \alpha_1 + \widetilde{\mu}_1 = \widetilde{\beta}\lambda_1 z'(\widetilde{U}_1).$$

This immediately implies $u_1 = U_1$, a contradiction, since MH_1 imposes $u_1 < U_1$. Thus, MH_1 is binding and $\widetilde{U}_1 = \widetilde{u}_1 + K_1$. From FOC2c, and $\widetilde{\mu}_1 \geq 0$, we derive

$$\widetilde{\mu}_1 = \frac{P_2}{P_1}\left[\alpha_2 - \widetilde{\beta}\lambda_2 z'(\widetilde{U}_2)\right] \geq 0. \tag{A6}$$

From FOC1c, we derive,

$$\widetilde{\mu}_1 + \frac{\widetilde{\delta}}{P_1} = \widetilde{\beta}\lambda_1 z'(\widetilde{U}_1) - \alpha_1 \geq 0. \tag{B6}$$

$\overline{IC}_i$ constraints and MH_1 impose $\widetilde{U}_2 > \widetilde{U}_1 > \widetilde{u}_1 > \widetilde{u}_2$ and therefore, combining A6 and B6, we obtain,

$$\frac{\alpha_2}{\lambda_2} \geq \widetilde{\beta} z'(\widetilde{U}_2) > \widetilde{\beta} z'(\widetilde{U}_1) \geq \frac{\alpha_1}{\lambda_1}.$$

It follows that we must have $\alpha_2(1-\lambda_2) > (1-\alpha_2)\lambda_2$, or $\alpha_2 > \lambda_2$.

Combining A6 and B6 again, assuming that $\widetilde{\delta} > 0$ for a while, we obtain

$$\widetilde{\delta} = \lambda_1 P_1[\widetilde{\beta} z'(\widetilde{U}_1) - (\alpha_1/\lambda_1)] + P_2\lambda_2[\widetilde{\beta} z'(\widetilde{U}_2) - (\alpha_2/\lambda_2)] > 0,$$

or equivalently,

$$\lambda_1 P_1 z'(\widetilde{U}_1) + \lambda_2 P_2 z'(\widetilde{U}_2) > \frac{P_\alpha}{\widetilde{\beta}}. \tag{C6}$$

The allocation is determined by a system of four equations, the first three are obviously $\overline{IC}_1$, MH_1 and $\overline{RC}$, expressed as equalities. To find the fourth equation, we eliminate Lagrange multipliers from FOC1c, FOC3c and FOC4c. More precisely, adding FOC1c and FOC3c yields

$$\widetilde{\mu}_1 = \lambda_1\widetilde{\beta}[P_1 z'(\widetilde{u}_1 + K_1) + (1-P_1)z'(\widetilde{u}_1)] - 1 + \alpha_2, \tag{D6}$$

so that $\widetilde{\mu}_1 \geq 0$ if α_2 is large enough given the allocation. On the other hand, substituting A6 yields,

$$\frac{P_2}{P_1}\left[\alpha_2 - \widetilde{\beta}\lambda_2 z'(\widetilde{U}_2)\right] = \lambda_1\widetilde{\beta}[P_1 z'(\widetilde{u}_1 + K_1) + (1-P_1)z'(\widetilde{u}_1)] - (1-\alpha_2),$$

or

$$\frac{P_\alpha}{\widetilde{\beta}} = \lambda_2 P_2 z'(\widetilde{U}_2) + \lambda_1 P_1[P_1 z'(\widetilde{u}_1 + K_1) + (1 - P_1) z'(\widetilde{u}_1)]. \tag{E6}$$

Note that if E6 holds, since $z'(\widetilde{U}_1) > z'(\widetilde{u}_1)$, then, necessarily, C6 holds, and we conclude that $\widetilde{\delta} > 0$. Substituting the expression for $\widetilde{\beta}$ derived above, and rearranging terms yields the fourth equation that we need to solve the problem,

$$\frac{\lambda_2}{\lambda_1 \alpha_2} \frac{[P_2(1 - P_\alpha) z'(\widetilde{U}_2) - P_\alpha(1 - P_2) z'(\widetilde{u}_2)]}{(P_2 - P_1)} = P_1 z'(\widetilde{u}_1 + K_1) + (1 - P_1) z'(\widetilde{u}_1) \tag{F}$$

The second-best optimum $((\widetilde{U}_1, \widetilde{u}_1), (\widetilde{U}_2, \widetilde{u}_2))$ is fully determined by $\overline{RC}$, $MH_1, \overline{IC}_1$ expressed as equalities and condition F. The condition $\widetilde{\mu}_1 \geq 0$ yields, with the expression for $\widetilde{\beta}$, a lower bound on the values of α_2 that can be derived from A6, that is, equivalently, from $\alpha_2 \geq \lambda_2 \widetilde{\beta} z'(\widetilde{U}_2)$.

Q.E.D.

The second-best optimal solutions that we get in the case of moral hazard and adverse selection combined are analogous to solutions derived in the case of nonnegative repayment constraints. If the social weights of types are close to their true frequency in the population, the solution exhibits *equal treatment* and *incomplete insurance*. Both types obtain the same payment in the event of "success" as well as in the event of "failure" and incomplete insurance takes care of effort incentives. The second-best optimum is a *separating* allocation à la Rothschild-Stiglitz only when the weight of the talented types is sufficiently higher than their frequency in the population. In other words, to get a separating optimum, the social planner must be willing to markedly favor the highly productive types. These allocations are trivially not first-best efficient since, in the absence of nonnegative repayment constraints, first-best efficiency requires full insurance. In a certain sense, the effort incentive constraints MH, by imposing incomplete insurance, play the same rôle as nonnegative repayment constraints in Section 3 above. In fact, it may happen that the negative repayment constraint also plays the rôle of an effort incentive constraint: if the difference between the payoffs in the two states is sufficiently large, the agents will prefer to exert the high level of effort. It follows that if we combine nonnegative repayment constraints with moral hazard, in many typical cases, one of the MH or the NR constraints will be redundant. Our study of the moral hazard case confirms the intuition that the second-best optimal solution exhibits *equal treatment*, when the social weights are sufficiently close to the empirical frequencies of types, that is, in the neighborhood of the standard utilitarian optimum. The solutions entail a form of *exploitation of the talented*, by means of cross-subsidies between types, since the less talented are also producing less surplus per capita. This subsidy from the talented survives as a price paid to solve the incentive problem, even if the social weight of the talented is increased. It is only when the social welfare function sufficiently favors the talented that the incentive problem is solved by means of *screening*, imposing a higher level of risk (and return) on the most productive agents.

5 Conclusion

We have studied optimal student-loan contracts in a simple economy with private information. There are two unobservable types of students with different probability distributions of individual labour-market outcomes (adverse selection). In addition, students choose an effort variable, affecting the probabilities of success, that is not observed by the lender (moral hazard). Students are also risk-averse, leading to an optimal insurance problem. We have described the set of second-best optimal (or interim efficient) incentive-compatible menus of loan contracts. There are two types of optima: the *separating* and *equal treatment* allocations. Equal treatment arises when the social weights of types are in the neighborhood of their frequencies in the student population. In this case, the *ex post* student payoffs are the same for all types as a function of individual outcomes. Students are *ex ante* unequal since they differ in their probability of success on the labour market. This type of allocation is different form the familiar menus of separating contracts in a screening model à la Rothschild-Stiglitz. The separating menus, in which the talented students bear more risk than the less-talented ones, appear only if the social weight of talented types is sufficiently greater than the latter type's frequency. In both cases, the optimal menus of contracts exhibit incomplete insurance, as a consequence of moral hazard; they typically involve cross-subsidies in favour of the less-talented. The less-talented obtain the maximal amount of insurance, compatible with effort incentives. Optimal student loans are always income-contingent. If the student-loan contracts are interpreted as a form of graduate tax, this tax is progressive.

6 References

Arnott, Richard J. (1991), “Moral Hazard and Competitive Insurance Markets", in Georges Dionne. ed.,*Contributions to Insurance Economics*, Kluwer Academic Publishers, Dordrecht, the Netherlands.

Baron, David P. and David Besanko (1987), “Monitoring, Moral Hazard, Asymmetric Information, and Risk Sharing in Procurement Contracting", *Rand Journal of Economics*, **18**, 509-532.

Barr, Nicholas and Alison Johnston (2010), “Interest Subsidies on Student Loans: A Better Class of Drain", CEE DP 114, *London School of Economics*, London, UK.

Bester, Helmut (1985), “Screening vs. Rationing in Credit Markets with Imperfect Information", *American Economic Review*, **75**, 850-855.

Caillaud, Bernard, Guesnerie, Roger and Patrick Rey (1992), "Noisy Observations in Adverse Selection Models", *Review of Economic Studies*, **59**, 596-615.

Carneiro, Pedro and James J. Heckman (2002), “The Evidence on Credit Constraints in Post-secondary Schooling", *Economic Journal*, **112**, 705-734.

Chassagnon, Arnold and Pierre-André Chiappori (1997), “Insurance under Moral Hazard and Adverse Selection: The Case of Pure Competition", *Paris School of Economics, manuscript*, http://umrpse.ens.fr/users/chassagnon/pub.html.

Chatterjee, Satyajit and Felicia Ionescu (2011),“Insuring Student Loans against the Financial Risk of College Failure", *manuscript*, *Colgate University*, Hamilton, New York.

Crocker, Keith J. and Arthur Snow (1985), “The Efficiency of Competitive Equilibria in Insurance Markets with Asymmetric Information", *Journal of Public Economics*, **26**, 207-219.18, No. 3 (Autumn, 2005), pp. 1021-1073

Dionne, Georges and Pierre Lasserre (1985), "Dealing with Moral Hazard and Adverse Selection Simultaneously", unpublished working paper no 8559*, Centre de Recherche sur les Transports, Université de Montréal*, Montreal, Canada.

Dionne, Georges, Doherty, Neil and Nathalie Fombaron (2000), "Adverse Selection in Insurance Markets", Chapter 7 in Georges Dionne. ed.*, Handbook of Insurance*, Kluwer Academic Publishers, Dordrecht, Netherlands.

Faynzilberg, Peter S. and Praveen Kumar (2000), "On the Generalized Principal-Agent Problem: Decomposition and Existence Results", *Review of Economic Design*, **5**, 23-58.

Freixas, Xavier and Jean-Charles Rochet (1998), *Microeconomics of Banking*, MIT Press, Cambridge, Massachusetts.

Hellwig, Martin F. (1987), "Some Recent Developments in the Theory of Competition in Markets with Adverse Selection", *European Economic Review*, **31**, 319-325.

Henriet, Dominique and Jean-Charles Rochet (1990), "Efficiency of Market Equilibria with Adverse Selection: The Case of Insurance", Chapter 9 in Paul Champsaur *et al.* eds.*, Essays in Honor of Edmond Malinvaud*, *volume 1*, MIT Press, Cambridge, Massachusetts.

Ionescu, Felicia (2009),"The Federal Student Loan Program: Quantitative Implications for College Enrollment and Default Rates", *Review of Economic Dynamics*, **12**, 205-231.

Jacobs, Bas and Frederick Van der Ploeg (2006),"Guide to Reform of Higher Education: A European Perspective", *Economic Policy*, **21(47)**, 535-592.

Keane, Michael and Kenneth Wolpin (2001), "The Effect of Parental Transfers and Borrowing Constraints on Educational Attainment", *International Economic Review*, **42**, 1051-1103.

Laffont Jean-Jacques and David Martimort (2002), *The Theory of Incentives: The Principal-Agent Model*, Princeton University Press, Princeton, New Jersey.

Laffont Jean-Jacques and Jean-Charles Rochet (1998), "Regulation of a Risk-Averse Firm", *Games and Economic Behavior*, **25**, 149-173.

Lochner Lance, J. and Alexander Monge-Naranjo (2010), “The Nature of Credit Constraints and Human Capital", *manuscript, Pennsylvania State University*, forthcoming, AER.

Lochner Lance, J. and Alexander Monge-Naranjo (2011), “Credit Constraints in Education", *NBER Working Paper 17435,* NBER*,* Cambridge, Massachusetts.

McAfee, R. Preston J. and John McMillan (1986), “Bidding for Contracts: A Principal-Agent Analysis", *Rand Journal of Economics*, **17**, 326-338.

Miyazaki, Hajime (1977), “The Rat Race and Internal Labor Markets", *Bell Journal of Economics*, **8(2)**, 394-418.

Myerson, Roger B. (1982), “Optimal Coordination Mechanisms in Generalized Principal-Agent Models", *Journal of Mathematical Economics*, **10**, 67-81.

Picard, Pierre (1987), "On the Design of Incentive Schemes under Moral Hazard and Adverse Selection", *Journal of Public Economics*, **33**, 305-331.

Rothschild, Michael and Joseph E. Stiglitz (1976), “Equilibrium in Competitive Insurance Markets", *Quarterly Journal of Economics*, **90**, 629-650.

Spence, Michael (1978), “Product Differentiation and Performance in Insurance Markets", *Journal of Public Economics*, **10**, 427-447.

Stinebrickner, Todd and Ralph Stinebrickner (2008), “The Effect of Credit Constraints on the College Drop-out Decision: A Direct Approach Using a New Panel Study", *American Economic Review*, **98**, 2163-2184.

Sung Jaeyoung (2005), “Optimal Contracts under Adverse Selection and Moral Hazard: A Continuous-Time Approach", *Review of Financial Studies* , **18(3)**, 1021-1073.

Wilson, Charles (1977), “A Model of Insurance Markets with Incomplete Information", *Journal of Economic Theory*, **16**, 167-207.

7 Appendix: Proofs

Proof of Lemma 1: Assume, by way of contradiction, that IC_1 holds strictly and that IC_2 is binding. Then $\mu_1 = 0$. From FOC2 and $\mu_2 \geq 0$, we get

$$\mu_2 = \beta_2 z'(U_2) - \alpha_2 \geq 0,$$

and since $\beta_2 z'(U_2) \geq \alpha_2 > 0$ we find $\beta_2 > 0$: that is, RC_2 is binding.

From FOC2 and FOC4, we easily derive,

$$\beta_2 z'(u_2) = \alpha_2 + \mu_2 = \beta_2 z'(U_2),$$

and therefore, $U_2 = u_2$. IC constraints then imply

$$u_1 > U_2 = u_2 > U_1.$$

Adding FOC1 and FOC3, we obtain

$$\beta_1 \left[p_1 z'(U_1) + (1 - p_1) z'(u_1) \right] = \alpha_1 - \mu_2.$$

But from FOC1 we derive $\mu_2 = (p_1/p_2)(\alpha_1 - \beta_1 z'(U_1))$. Substituting this expression for μ_2 in the above equality, we obtain,

$$\beta_1 \left[\left(p_1 - \frac{p_1}{p_2} \right) z'(U_1) + (1 - p_1) z'(u_1) \right] = \alpha_1 \left(1 - \frac{p_1}{p_2} \right) > 0.$$

Now, since $u_1 > U_1$, we have

$$\left(p_1 - \frac{p_1}{p_2} \right) z'(U_1) + (1 - p_1) z'(u_1) > \left(p_1 - \frac{p_1}{p_2} + 1 - p_1 \right) z'(U_1) = \left(1 - \frac{p_1}{p_2} \right) z'(U_1) > 0,$$

since $p_2 > p_1$. This proves that $\beta_1 > 0$ and therefore, RC_1 is binding. From RC_2, using the fact that $U_2 = u_2$, we get,

$$B_2(q_2^*) = z(u_2), \tag{32}$$

and from RC_1 we get

$$p_1 z(U_1) + (1 - p_1) z(u_1) = B_1(q_1^*). \tag{33}$$

It now follows from $B_2(q_2^*) > B_1(q_1^*)$ that

$$p_1 z(U_1) + (1 - p_1) z(u_1) < z(u_2). \tag{A1}$$

Now, since $z(.)$ is convex,

$$p_1 z(U_1) + (1 - p_1) z(u_1) > z(p_1 U_1 + (1 - p_1) u_1),$$

and since $z(.)$ is increasing, $u_1 > U_1$ and $p_2 > p_1$,

$$z(p_1 U_1 + (1 - p_1) u_1) > z(p_2 U_1 + (1 - p_2) u_1).$$

By assumption, IC_2 is binding and since $U_2 = u_2$, we have $p_2 U_1 + (1 - p_2) u_1 = u_2$. from this and the above inequalities we derive

$$p_1 z(U_1) + (1 - p_1) z(u_1) > z(u_2), \tag{B1}$$

a contradiction, since inequality $A1$ directly contradicts $B1$. *Q.E.D.*

Proof of Proposition 1: By Lemma 1, we know that IC_1 is binding. There are two possible cases : either IC_2 is slack, or IC_2 is binding too, at the second-best optimum.

Step 1: Assume first that IC_1 is binding and IC_2 holds as a strict inequality. This implies $\mu_2 = 0$, and by FOC1,

$$\mu_1 = \beta_1 z'(U_1) - \alpha_1 \geq 0.$$

As a consequence, we find that $\beta_1 > 0$ and that RC_1 is binding. Now, FOC 1 and FOC3 combined yield

$$\beta_1 z'(u_1) = \beta_1 z'(U_1), \tag{34}$$

and therefore $U_1^{**} = u_1^{**}$. The IC constraints then imply

$$U_2^{**} > U_1^{**} = u_1^{**} > u_2^{**}. \tag{35}$$

Since RC_1 is binding, we have $z(U_1^{**}) = B_1(q_1^*)$ which is equivalent to $U_1^{**} = u(B_1(q_1^*))$, so that the allocation is first-best optimal for type 1. Adding FOC1 and FOC2 on the one hand, and FOC3 and FOC4 on the other, yields the linear system,

$$\begin{aligned} p_\alpha &= \beta_1 p_1 z'(U_1^{**}) + \beta_2 p_2 z'(U_2^{**}) \\ 1 - p_\alpha &= \beta_1 (1 - p_1) z'(U_1^{**}) + \beta_2 (1 - p_2) z'(u_2^{**}), \end{aligned} \tag{36}$$

where by definition, $p_\alpha = \alpha_1 p_1 + \alpha_2 p_2$. Given that $\alpha_1 + \alpha_2 = 1$, we have $p_1 < p_\alpha < p_2$. The above system has a non-zero determinant and a unique solution $(\beta_1^{**}, \beta_2^{**})$. It is easy to check that,

$$\beta_1^{**} = \frac{(1-p_\alpha)p_2 z'(U_2^{**}) - p_\alpha(1-p_2)z'(u_2^{**})}{z'(U_1^{**})[(1-p_1)p_2 z'(U_2^{**}) - p_1(1-p_2)z'(u_2^{**})]}; \tag{37}$$

$$\beta_2^{**} = \frac{p_\alpha - p_1}{[(1-p_1)p_2 z'(U_2^{**}) - p_1(1-p_2)z'(u_2^{**})]}. \tag{38}$$

Since $p_1 < p_\alpha < p_2$ and $z'(U_2^{**}) > z'(u_2^{**})$ both the numerators and the denominators of the above expressions are positive. We conclude that $\beta_2^{**} > 0$ and that RC_2 is binding. The utility levels (U_2^{**}, u_2^{**}) are fully determined by the intersection of RC_2 and IC_1. Using the expression for β_1^{**}, we find the optimal value of $\mu_1^{**} = \beta_1^{**} z'(U_1^{**}) - \alpha_1$, that is,

$$\mu_1^{**} = \frac{(1-p_\alpha)p_2 z'(U_2^{**}) - p_\alpha(1-p_2)z'(u_2^{**})}{(1-p_1)p_2 z'(U_2^{**}) - p_1(1-p_2)z'(u_2^{**})} - \alpha_1. \tag{39}$$

Now, we can check that this multiplier is always positive as follows. Rearranging terms, we easily find that $\mu_1^{**} \geq 0$ if and only if

$$[(1-p_\alpha)p_2 - \alpha_1(1-p_1)p_2]\, z'(U_2^{**}) \geq [p_\alpha(1-p_2) - \alpha_1 p_1(1-p_2)]\, z'(u_2^{**}),$$

but this is always true, since $z'(U_2^{**}) > z'(u_2^{**})$ and $(1-p_\alpha)p_2 - \alpha_1(1-p_1)p_2 = p_\alpha(1-p_2) - \alpha_1 p_1(1-p_2)$. The Kuhn and Tucker conditions being necessary and sufficient conditions for optimality here, we have found the second-best optimum.

Step 2: We now check that the allocation in which both IC constraints are binding is not an optimum. If IC_1 and IC_2 are simultaneously binding, then by Result 2a, $u_1 = u_2 = \widehat{u}$ and $U_1 = U_2 = \widehat{U}$. Adding FOC1 and FOC2 on the one hand, and FOC3 and FOC4 on the other hand, we find β_1 and β_2 as a function of (U, u). It is easy to check that,

$$\beta_1 = \frac{(1-p_\alpha)p_2 z'(\widehat{U}) - (1-p_2)p_\alpha z'(\widehat{u})}{z'(\widehat{u})z'(\widehat{U})(p_2-p_1)},$$

$$\beta_2 = \frac{(1-p_1)p_\alpha z'(\widehat{u}) - (1-p_\alpha)p_1 z'(\widehat{U})}{z'(\widehat{u})z'(\widehat{U})(p_2-p_1)}.$$

If one of these multipliers is negative, $(\widehat{U}, \widehat{u})$ cannot be an optimal solution. Assume then that these multipliers are both positive and that RC constraints are binding: $(\widehat{U}, \widehat{u})$ must solve the following system,

$$\begin{aligned} B_1^* &= p_1 z(\widehat{U}) + (1-p_1) z(\widehat{u}), \\ B_2^* &= p_2 z(\widehat{U}) + (1-p_2) z(\widehat{u}). \end{aligned}$$

It is not difficult to check that the solution is unique. The above system is linear in $(z(u), z(U))$ and has a nonzero determinant. Since z is a monotonically increasing function, it follows that $(\widehat{U}, \widehat{u})$ is uniquely determined. The intersection of RC_1 and RC_2 is non-empty. In addition, since $p_2 > p_1$ and $B_2^* > B_1^*$, it is easy to check that we must have $\widehat{U} > \widehat{u}$.

We now show that $(\widehat{U}, \widehat{u})$ is Pareto-dominated by the second-best solution derived in *Step 1* above, that is, the allocation (U_1^*, u_1^*) where $U_1^* = u_1^*$ is identical to the first best for type 1, the second-best allocation (U_2^{**}, u_2^{**}), with $U_2^{**} > u_2^{**}$, such that IC_1 is binding, for type 2.

Since it maximizes $p_1 U_1 + (1-p_1) u_1$ subject to RC_1, it is obvious that (U_1^*, u_1^*) dominates $(\widehat{U}, \widehat{u})$ for this type. Consider now type 2. Define the function,

$$\varphi_i(u) = u\left(\frac{B_i^* - (1-p_i) z(u)}{p_i}\right),$$

and remark that (U, u) satisfies RC_i as an equality if and only if $U = \varphi_i(u)$. This function is just RC_i in the (u, U) plane; it is a monotonically decreasing and strictly concave function of u, and $(U_1^*, u_1^*) = (u_1^*, u_1^*)$ maximizes $p_1 U + (1-p_1) u$ subject to $U = \varphi_1(u)$. By definition, $\varphi_2(\widehat{u}) - \varphi_1(\widehat{u}) = 0$. Since φ_1 is decreasing and intersects the diagonal $U = u$ only once at point u_1^*, we also have $\widehat{u} < u_1^*$. Define the indifference curve,

$$U = \psi_i(u) = \frac{u_1^* - (1-p_i) u}{p_i},$$

and remark that (U_2, u_2) satisfies IC_1 as an equality if and only if $U_2 = \psi_1(u_2)$. Since φ_1 is a concave function, the locus of points (U, u) satisfying RC_1 is entirely included in the half-plane $p_1 U + (1-p_1) u \leq u_1^*$ or more precisely, $U = \varphi_1(u)$ implies $U \leq \psi_1(u)$, with a strict inequality if $u \neq u_1^*$. It follows that $\varphi_2(\widehat{u}) - \psi_1(\widehat{u}) = \varphi_1(\widehat{u}) - \psi_1(\widehat{u}) < 0$. Now, since u_1^* maximizes the utility of type 1 subject to RC_1, we have $\psi_1(u_1^*) = \varphi_1(u_1^*) = U_1^* = u_1^*$ and since type 2 is more productive, *i.e.*, $B_2^* > B_1^*$ and $p_2 > p_1$, we can write, $\varphi_2(u_1^*) - \psi_1(u_1^*) =$

$\varphi_2(u_1^*)-\varphi_1(u_1^*) > 0$. It is not difficult to check that the mapping $(\varphi_2(u)-\psi_1(u))$ is continuous and strictly decreasing on the interval $[\widehat{u}, u_1^*]$. By the Intermediate Value Theorem, there exists a point u in the open interval $(\widehat{u}, u_1^*)$ such that $\varphi_2(u) - \psi_1(u) = 0$, and this point is unique. But we know that this point is precisely u_2^{**}, since by definition (U_2^{**}, u_2^{**}) is the intersection of RC_2 and IC_1, and thus we have $\varphi_2(u_2^{**}) - \psi_1(u_2^{**}) = 0$. Now, the expected utility of type 2, that is, $p_2\varphi_2(u) + (1 - p_2)u$, is a strictly concave and strictly increasing function of u on the interval $[\widehat{u}, u_2^*]$, where u_2^* is the first-best, full-insurance solution for type 2. We conclude that $p_2\widehat{U} + (1 - p_2)\widehat{u} < p_2U_2^{**} + (1 - p_2)u_2^{**}$: the allocation $(\widehat{U}, \widehat{u})$ is dominated by (U_2^{**}, u_2^{**}) for type 2. The solution $((u_1^*, u_1^*), (U_2^{**}, u_2^{**}))$ defined above is the only second-best optimum.
Q.E.D.

Proof of Proposition 3. From Result 3 we know that if IC_2 is binding, then, both IC constraints are binding. Therefore, we consider a solution such that IC_1 is binding and IC_2 is slack. If IC_2 is slack, then $\mu_2 = 0$. The first-order conditions become,

$$\alpha_1 + \mu_1 = \beta\lambda_1 z'(U_1); \tag{FOC1b}$$

$$\alpha_2 p_2 - \mu_1 p_1 = \beta\lambda_2 p_2 z'(U_2); \tag{FOC2b}$$

$$\alpha_1 + \mu_1 = \beta\lambda_1 z'(u_1) + \delta_1; \tag{FOC3b}$$

$$\alpha_2(1 - p_2) - \mu_1(1 - p_1) = \beta\lambda_2(1 - p_2)z'(u_2); \tag{FOC4b}$$

where $\delta_1 = \delta/(1 - p_1) \geq 0$. Equation FOC1b immediately implies $\beta^{**} > 0$ since $z' > 0$, $\alpha_1 + \mu_1 > 0$, and we easily derive (using FOC1b and FOC2b),

$$\beta^{**} = \frac{\alpha_1 p_1 + \alpha_2 p_2}{\Sigma_i \lambda_i p_i z'(U_i^{**})}. \tag{40}$$

Hence, $\overline{RC}$ is binding. FOC1b and FOC3b immediately show that $\delta_1 > 0$ implies $U_1^{**} > u_1^{**}$, and, in addition, since $\delta(u_0 - u_1) = 0$, we know that $\delta^{**} > 0$ implies $u_1^{**} = u_0$. But if $U_1 = u_1$ and $\delta = 0$, we must have $U_1 = u_1 \leq u_0$ and therefore, IR_1 cannot be satisfied as a strict inequality. Thus, $\delta^{**} > 0$, and since IC constraints hold, we know that, necessarily,

$$U_2^{**} > U_1^{**} > u_0 = u_1^{**} > u_2^{**}.$$

Now, from FOC1b and FOC2b, since μ_1^{**} must be nonnegative, and since $U_2^{**} > U_1^{**}$, we derive,

$$\frac{\alpha_2}{\lambda_2} \geq \beta^{**} z'(U_2^{**}) > \beta^{**} z'(U_1^{**}) \geq \frac{\alpha_1}{\lambda_1}, \tag{41}$$

and therefore, we must have $\alpha_2/\lambda_2 > \alpha_1/\lambda_1$. Given that $\lambda_1 + \lambda_2 = 1$ and $\alpha_1 + \alpha_2 = 1$, this is equivalent to

$$\alpha_2 > \lambda_2. \tag{42}$$

It follows that this type of allocation (or solution-candidate) is a second-best optimum only if $\alpha_2 > \lambda_2$.

Adding FOC2b and FOC4b, we find the following condition,

$$\frac{\alpha_2}{\lambda_2} = \beta^{**} \frac{[p_2(1-p_1) z'(U_2^{**}) - p_1(1-p_2) z'(u_2^{**})]}{[p_2 - p_1]}. \tag{C}$$

The second-best optimality problem is convex, due to the assumed strict convexity of z. It follows that the FOCs are necessary and sufficient. The optimal allocation (four unknowns here) is thus obtained as a solution of conditions C, $\overline{RC}$ and IC_1 and $u_1^{**} = u_0$ (four equations).

Q.E.D.

Proof of Proposition 4. If $u_1^{**} = u_2^{**} = u_0$, IC constraints are both binding and $U_1^{**} = U_2^{**} = U^{**}$. Adding FOC1 and FOC2 together then yields,

$$\beta^{**} = \frac{p_\alpha}{p_\lambda z'(U^{**})},$$

where by definition $p_\lambda = p_1\lambda_1 + p_2\lambda_2$ and $p_\alpha = p_1\alpha_1 + p_2\alpha_2$. It follows from $\beta^{**} > 0$ that $\overline{RC}$ is binding. Adding then FOC3 and FOC4 yields

$$\delta^{**} = (1 - p_\alpha) - \beta^{**} z'(u_0)(1 - p_\lambda),$$

or, substituting the value of β^{**}, we obtain

$$\delta^{**} = (1 - p_\alpha) - (1 - p_\lambda)\frac{p_\alpha z'(u_0)}{p_\lambda z'(U^{**})}. \tag{43}$$

Thus, $\delta^{**} > 0$ (a required condition here) if and only if

$$\frac{(1 - p_\alpha)p_\lambda}{(1 - p_\lambda)p_\alpha} > \frac{z'(u_0)}{z'(U^{**})}. \tag{44}$$

We know that $u_0 < U^{**}$ and thus $z'(u_0) < z'(U^{**})$ for otherwise, IR constraints could not be satisfied as strict inequalities. On the other hand, since $p_1 < p_2$, we have $p_\alpha \le p_\lambda$ if and only if $\alpha_2 \le \lambda_2$. It follows that the above inequality will always be true if $\alpha_2 \le \lambda_2$. The inequality is true also if α_2 greater than but close enough to λ_2.

The binding resource constraint yields,

$$B^* = \Sigma_i \lambda_i [p_i z(U_i^{**}) + (1-p_i) z(u_0)] = p_\lambda z(U^{**}) + (1-p_\lambda) w_0, \tag{45}$$

since $u_0 = u(w_0)$. From this expression, and from $z(U^{**}) = w(q_i^*) - R_i^{**}$ for all i and the definition of B^*, we easily derive,

$$\Sigma_i \lambda_i p_i R_i^{**} = \Sigma_i \lambda_i \gamma_i q_i^*, \tag{46}$$

and

$$w(q_1^*) - R_1^{**} = w(q_2^*) - R_2^{**}. \tag{47}$$

These equations fully determine U^{**}. (Note that there are conditions on w and γ_i that would ensure $R_i^{**} \ge 0$ here; otherwise, we would get a corner solution and $\overline{NR}_1$ would be binding.)

We finally need to find the conditions under which the multipliers μ_1 and μ_2 are nonnegative (this is required for a solution candidate to be admissible). Adding FOC2 and FOC4 and rewriting FOC1 yields the linear system,

$$\begin{aligned} \mu_1^{**} p_1 - \mu_2^{**} p_2 &= \beta^{**} \lambda_1 p_1 z'(U^{**}) - \alpha_1 p_1; && \text{(FOC1)} \\ -\mu_1^{**} + \mu_2^{**} &= \beta^{**} \lambda_2 [p_2 z'(U^{**}) + (1-p_2) z'(u_0)] - \alpha_2. && \text{(FOC4+2)} \end{aligned}$$

Recall that $\beta^{**} z'(U^{**}) = p_\alpha / p_\lambda$. This system has a nonzero determinant, $p_1 - p_2$, and a unique solution (μ_1^{**}, μ_2^{**}). More precisely, since $u_0 < U^{**}$, we find

$$\mu_1^{**} = \frac{p_\alpha \lambda_2 p_2 (1-p_2)}{p_\lambda (p_2 - p_1)} \left(1 - \frac{z'(u_0)}{z'(U^{**})}\right) > 0. \tag{48}$$

It is not difficult to check that μ_1^{**} is positive: this tells us that IC_1 is always binding. We compute the solution μ_2^{**} as follows.

$$\mu_2^{**} = \frac{p_1}{(p_2 - p_1) p_\lambda} \left[(p_\lambda - p_\alpha) + p_\alpha \lambda_2 (1-p_2) \left(1 - \frac{z'(u_0)}{z'(U^{**})}\right) \right]. \tag{49}$$

It is now easy to check that we have $\mu_2^{**} > 0$ if $p_\alpha \leq p_\lambda$. But is is also true that $\mu_2^{**} > 0$ for a range of values of α_2 that are greater than, but close enough to λ_2, since $z'(u_0) < z'(U^{**})$. (We skip the derivation of the threshold value here). We conclude that the egalitarian, maximal insurance solution is the second-best optimum for $\alpha_2 \leq \lambda_2$ and for values of α_2 that are greater than, but close enough to λ_2.

Q.E.D.