Mechanism Design in an Atkeson-Lucas Environment with Capital: A Progress Report

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University of Chicago / MFI Presentation
2010.05.13

Physical environment

Agents	$(\mathbb{I}, \mathcal{B}_{\mathbb{I}}, N)$
States of nature	$(\Omega, \mathcal{B}_{\Omega}, P)$
Time	$\mathbb{T} = \{0, 1, 2, \ldots\}$.
Goods	Composite commodity in each (ω, t)
Individual shocks	$\Theta \subseteq \mathbb{R}_{+}$, $\theta : \mathbb{I} \times \Omega \times \mathbb{T} \to \Theta$
	$\bar{p} \in \text{int}(\Delta(\Theta))$, $\theta(i, \cdot, t) \sim \bar{p}$
	IID across agents and time periods, "LLN"
Allocation	$\alpha : \mathbb{I} \times \Omega \times \mathbb{T} \to \mathbb{R}_{+}$
Utility	$U_i(\alpha) = \int_{\Omega} \sum_{t \in \mathbb{T}} \beta^t \alpha(i, \omega, t)^{\gamma} / \gamma \, dP(\omega)$;
	$\gamma \leq -1$
Capital	$\kappa_{\alpha} : \Omega \times \mathbb{T} \to \mathbb{R}_{+}$;
	$\kappa_{\alpha}(\omega, 0) = \bar{k}$ ¿ 0;
	$\kappa_{\alpha}(\omega, t+1) = R\kappa_{\alpha}(\omega, t) - \int_{\mathbb{I}} \alpha(i, \omega, t) \, dN(i)$
Feasibility	For all t, $\kappa_{\alpha}(\cdot, t) \geq 0$ a.s.

Why add capital?

Make the model more applicable?

More simply characterize the efficient IC allocation.

Bring implementation issues into focus.

- Is the equilibrium of an allocation mechanism unique?
  - Non-uniqueness can resemble "run equilibrium" in the D-D model

Notational conventions

If $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ are measurable spaces,

- X^Y denotes the space of measurable functions $\vec{x} : Y \to X$
- $\vec{X}$ denotes X^Y when Y is clear from context
  - Typically the domain is $\mathbb{I}$ or $\mathbb{T}$
- $\vec{x}$ denotes an element of $\vec{X}$

If $\vec{X} = X^{\mathbb{T}}$, then

- $\text{proj}_t^X(\vec{x}) = (\vec{x}_0, \ldots, \vec{x}_t)$
- $\text{proj}_t^X(\vec{X}) = X^{t+1}$

If $x = (y, z)$, then $\text{proj}_Y^X(x) = y$

Information — Agents and planner

At every (ω, t), i privately observes (later recalls) $\theta(i, \omega, t)$

At every date, i sends a message to the planner.

- Message must be measurable in agent's information
- Θ is the message space. (Direct mechanism)

After agents' messages have been received, the planner sends a message to all agents.

- Message must be measurable in planners' information
- Message space $\mathbb{M}$ is not restricted.

Allocation mechanism and reporting

A mechanism is $\mu = (\mu^{\mathbb{M}}, \mu^{\mathbb{A}})$

$\mu^{\mathbb{M}} = (\mu_t^{\mathbb{M}})_{t \in \mathbb{T}}$

$\mu_t^{\mathbb{M}} : (\Theta^{\mathbb{I}})^{t+1} \to \mathbb{M}$

$\mu^{\mathbb{A}} = (\mu_t^{\mathbb{A}})_{t \in \mathbb{T}}$

$\mu_t^{\mathbb{A}} : (\Theta^{\mathbb{I}})^{t+1} \to \mathbb{R}_+^{\mathbb{I}}$

An agent's reporting plan is $\rho = (\rho_t)_{t \in \mathbb{T}}$

$\rho_t : (\Theta \times \mathbb{M})^t \times \Theta \to \Theta$

A reporting profile is $\vec{\rho} = (\vec{\rho}_t)_{t \in \mathbb{T}}$

$\vec{\rho}_t : \mathbb{I} \times (\Theta \times \mathbb{M})^t \times \Theta \to \Theta$

Mechanism as a game form

$G(\mu, \vec{\rho}) = (G^{\mathbb{M}}(\mu, \vec{\rho}), G^{\mathbb{A}}(\mu, \vec{\rho}))$

$[G^{\mathbb{M}}(\mu, \vec{\rho})] = ([G_t^{\mathbb{M}}(\mu, \vec{\rho})])_{t \in \mathbb{T}}$

- $[G^{\mathbb{M}}(\mu, \vec{\rho})](\omega, t)$ is the recursively defined element of $\Theta^{\mathbb{I}} \times \mathbb{M}$ that results in (ω, t) when agents report their shocks according to $\vec{\rho}$ and the planner bases messages on those report profiles.

$G^{\mathbb{A}}(\mu, \vec{\rho}) \in \mathbb{A}$

- $[G^{\mathbb{A}}(\mu, \vec{\rho})](i, \omega, t)$ is the consumption that the planner allocates to i in (ω, t) on the basis of messages that are sent between agents and the planner, as specified by $G^{\mathbb{M}}(\mu, \vec{\rho})$

μ is feasible if, for all $\vec{\rho}$, $G^{\mathbb{A}}(\mu, \vec{\rho})$ is feasible.

Formal anonymity

Mechanism μ is formally anonymous if it always gives every agent consumption that is measurable with respect to his past and current reports and the planner's current message.

Formally, there is a function sequence $\alpha^* = (\alpha^*_t : \Theta^{t+1} \times \mathbb{M} \to \mathbb{R}_+)_{t \in \mathbb{T}}$ such that, for all i, ω, t, and $\vec{\rho}$,

$$\begin{aligned}[G^{\mathbb{A}}(\mu, \vec{\rho})](i, \omega, t) = \alpha^*_t([[\mathrm{proj}^{\Theta^{\mathbb{I}} \times \mathbb{M}}_{\Theta^{\mathbb{I}}} \circ G^{\mathbb{M}}(\mu, \vec{\rho})](\omega, 0)](i), \\ \ldots, [[\mathrm{proj}^{\Theta^{\mathbb{I}} \times \mathbb{M}}_{\Theta^{\mathbb{I}}} \circ G^{\mathbb{M}}(\mu, \vec{\rho})](\omega, t)](i), \\ [\mathrm{proj}^{\Theta^{\mathbb{I}} \times \mathbb{M}}_{\mathbb{M}} \circ G^{\mathbb{M}}(\mu, \vec{\rho})](\omega, t))\end{aligned}$$

Henceforth I consider formally anonymous mechanisms

It is innocuous that I have not assumed that agents recall their past consumption

Incentive compatibility

Allocation α is anonymous if all agents receive consumption as a function (the same for all agents) of past and current shocks, and the joint population distribution of those shocks.

For any anonymous α, let μ_α be the mechanism that treats profiles of reported shocks as α treats the actual shock profiles, and that specifies an uninformative planner's message.

α is incentive compatible if truthful revelation is a Bayesian Nash equilibrium of μ_α.

Even if α is feasible, it is possible for μ_α not to be feasible because $G^{\mathbb{A}}(\mu_\alpha, \vec{\rho})$ is an infeasible allocation for some untruthful reporting profile.

Mas-Colell and Vives (1993) show how, given an IC allocation, to construct a feasible mechanism, of which that is the equilibrium allocation of truthful reporting.

However, they do not show that the equilibrium is unique.

The optimal IC allocation

Construct a value function V for capital

Consider a set of agents with identical histories

Assume $\Theta = \{\lambda, \eta\}$, $0 < \lambda < \eta$

(WLOG, the entire population in period 0)

Optimally allocate k units of composite commodity among them, subject to IC

Agent reporting shock ϑ receives c_ϑ for consumption and k_ϑ for investment

Materials balance (MB) constraint binds:

$$p_\lambda(c_\lambda + k_\lambda) + p_\eta(c_\eta + k_\eta) = Rk$$

IC constraint for type λ binds:

$$\lambda c_\lambda^\gamma/\gamma + \beta V(k_\lambda) = \lambda c_\eta^\gamma/\gamma + \beta V(k_\eta)$$

Solve Lagrangean for: Maximize

$$p_\lambda[\lambda c_\lambda^\gamma/\gamma + \beta V(k_\lambda)] + p_\eta[\lambda c_\eta^\gamma/\gamma + \beta V(k_\eta)]$$

subject to MB and IC for λ

NB: Not convex constraints, but regular equality constraints

Result: $k_\vartheta = \phi_\vartheta c_\vartheta$

(ϕ_ϑ has a functional form, and $\phi_\lambda > \phi_\eta$)

Substitute into binding constraints:

$$p_\lambda c_\lambda(1 + \phi_\lambda) + p_\eta c_\eta(1 + \phi_\eta) = Rk$$

$$\lambda c_\lambda^\gamma/\gamma + \beta V(\phi_\lambda c_\lambda) = \lambda c_\eta^\gamma/\gamma + \beta V(\phi_h i c_\eta)$$

Represent $c_\vartheta = \xi_\vartheta k$ and suppose that $V(x) = vx^\gamma/\gamma$

$$[p_\lambda \xi_\lambda(1 + \phi_\lambda) + p_\eta \xi_\eta(1 + \phi_\eta)]k = Rk$$

$$[\lambda\xi_\lambda^\gamma + \beta v(\phi_\lambda\xi_\lambda)^\gamma]k^\gamma/\gamma = [\lambda\xi_\eta^\gamma + \beta v(\phi_h i\xi_\eta)^\gamma]k^\gamma/\gamma$$

Rewrite these equations as

$$p_\lambda\xi_\lambda(1+\phi_\lambda) + p_\eta\xi_\eta(1+\phi_\eta) = R$$

$$[\lambda + \beta v(\phi_\lambda)^\gamma)]\xi_\lambda^\gamma = [\lambda + \beta v(\phi_\eta)^\gamma]\xi_\eta^\gamma$$

$$[\lambda + \beta v(\phi_\lambda)^\gamma)]^{1/\gamma}\xi_\lambda = [\lambda + \beta v(\phi_\eta)^\gamma]^{1/\gamma}\xi_\eta$$

Solve (1) and (2) for ξ_λ and ξ_η

Solve

$$v = p_\lambda[\lambda + \beta v(\phi_\lambda^\gamma)]\xi_\lambda^\gamma + p_\eta[\lambda\xi_\eta^\gamma + \beta v(\phi_\eta^\gamma)]\xi_\eta^\gamma$$

for v

Infeasibility

Result: $c_\eta + k_\eta > c_\lambda + k_\lambda$

Recall MB constraint: $p_\lambda(c_\lambda + k_\lambda) + p_\eta(c_\eta + k_\eta) = Rk$

Let α be the optimal IC allocation

If $\vec{\rho}$ is the reporting profile in which all agents report receiving shock η, then $G^{\mathbb{A}}(\mu_\alpha, \vec{\rho})$ is not a feasible allocation

So μ_α is not a feasible mechanism

Some optimal mechanisms

In the optimal IC allocation, an agent's consumption is a function of his reported shocks and the capital stock.

Set $\mathbb{M} = \mathbb{R}_+$ and consider formally anonymous mechanisms in which the planner reports the current capital stock to agents at each (ω, t)

Several such mechanisms are feasible and partially implement the optimal IC allocation.

(A mechanism partially implements an allocation if that allocation is one of the equilibrium allocations of the mechanism)

Some optimal mechanisms

Let c_ϑ and k_ϑ be what the optimal IC allocation gives to agents who report shock ϑ in period 0

Suppose that $\tilde{p}_\eta = N(\{i \mid \rho_i \circ \theta(i, \omega, 0) = \eta\} > \bar{p}_\eta$

Give consumption c_ϑ to agents reporting shock ϑ

Set $\tilde{k}_\vartheta$ to satisfy

$$V(\tilde{k}_\lambda) - V(\tilde{k}_\eta) = V(k_\lambda) - V(k_\eta)$$

$$\tilde{p}_\lambda(c_\lambda + \tilde{k}_\lambda) + \tilde{p}_\lambda(c_\eta + \tilde{k}_\eta) = R\bar{k}$$

and continue recursively.

Truthful reporting is a perfect Bayesian equilibrium of this mechanism

For agents always to report shock η is also a PBE (Pan, 2008)

Some optimal mechanisms

Consider $R > 1$

In period 0, if $\tilde{p}_\vartheta > \bar{p}_\vartheta$, then give $\tilde{c}_\lambda$ and $\tilde{c}_\eta$ such that

$$\tilde{p}_\lambda \tilde{c}_\lambda + \tilde{p}_\eta \tilde{c}_\eta = (R-1)\bar{k}$$

$$\tilde{c}_\vartheta = 2\tilde{c}_{\vartheta'} \quad (\Theta = \{\vartheta, \vartheta'\})$$

In period 1, the capital stock will be $\bar{k}$

Give the optimal IC allocation from period 1 forward

Treat subsequent deviations similarly

Conjecture (almost certain): This mechanism implements (uniquely) the efficient IC allocation

Problem: The mechanism is not continuous

Some optimal mechanisms

Define $\phi = \frac{\bar{p}_\lambda(c_\lambda+k_\lambda)+\bar{p}_\eta(c_\eta+k_\eta)}{\tilde{p}_\lambda(c_\lambda+k_\lambda)+\tilde{p}_\eta(c_\eta+k_\eta)}$

Set $\tilde{c}_\vartheta = \phi c_\vartheta$ and $\tilde{k}_\vartheta = \phi k_\vartheta$

Recursively doing this weakly implements the optimal IC allocation (Mas-Colell and Vives, 1993)

Truthful revelation is not strictly dominant because IC constraint binds for λ

Change the model by making Θ an interval and $\bar{p}$ a measure with continuous, positive density

Now all IC constraints are strict

The mechanism fully implements the optimal IC allocation in strictly dominant strategies

A prima facie policy implication

Mechanisms that we know (or conjecture) to implement the optimal IC allocation, do not honor current-consumption promises when a positive-measure set of agents deviate from equilibrium

Several actual policies seem to enforce current-consumption promises

- Deposit-insurance payout funded by future, distortionary taxation
- Priority of short-maturity claims in bankruptcy
- Exclusion of payment-system obligations from bankruptcy

If failure of full implementation corresponds to financial fragility, then those policies may make the financial system fragile

Caveat: Rationales for those policies have not been modeled here